Dominant Firm and Competitive Bundling in Oligopoly Markets

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January 19, 2018

Abstract

This paper studies competitive bundling in an oligopoly market with one multi-product dominant firm and several symmetric small firms. In the model of competing against specialists each small firm produces a single product, while in the model of competing against generalists each small firm produces multiple products. In the model of competing against specialists, we show that the dominant firm will bundle if its dominance level is high, but will not bundle if the dominance level is low. In the model of competing against generalists, we show that the dominant firm's incentive to bundle differs from that of small firms. In particular, we find that (i) when the dominance level is low enough (and the number of firms is not too large), bundling hurts all firms and no firm bundles; (ii) when the dominance level is sufficiently high, bundling softens competition and all firms bundle; and (iii) when the dominance level is intermediate, the dominant firm bundles while small firms sell separately. Relative to previous literature, our paper offers a more complete analysis of the impacts of market structure (dominance level and the number of firms) on firms' incentives to bundle.

Keywords: Bundling, Tying, Dominance, Product Compatibility, Oligopoly

JEL Classification: D43, L13, L15

1 Introduction

A popular business strategy used by multi-product firms is to bundle their products. Anecdotal evidence suggests that in oligopoly markets, dominant firms rather than their small rivals often bundle their products. For example, Nespresso, the dominant firm in the espresso coffee market, sells brewing machines and capsules as a system. When its patents expired in 2012, which allowed its competitors to offer capsules and machines compatible with Nespresso's system, its parent company, Nestle, actively worked on ways to prevent competitors from doing this.[1]

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[1] Source: https://en.wikipedia.org/wiki/Nespresso.

Similarly, Apple Inc.—the prominent smartphone producer—requires that its Apple Watch series be coupled with its iPhone devices, while Android smartwatches, such as Samsung Gear, can be paired with any Android phone.

Many antitrust cases conform to the above pattern. For example, in *United States v. Microsoft*, Microsoft was alleged to abuse its dominance in the operating system market by bundling Internet Explorer with Windows, which hurts competing web browsers such as Netscape Navigator and Opera. Relatedly, Microsoft bundled Windows Media Player with Windows, and it was forced by antitrust authorities to offer an unbundled version of Windows in Europe and Korea. Bundling has since become an increasingly popular practice in the digital goods market. For instance, the giant search engine Google bundles all of its applications (on smartphones) in Google Play so that consumers must download them in an all-or-nothing fashion. Likewise, Microsoft requires that MS Office be sold only as a bundle.

The literature on competitive bundling (e.g, Matutes and Regibeau, 1988) typically studies symmetric firms in a duopoly model by adopting a two-dimensional Hotelling framework. Under this setting, the main conclusion is that relative to separate sales, bundling intensifies competition. Two recent papers advance our understanding of competitive bundling in two directions. Hurkens, Jeon, and Menicucci (2016, HJM henceforth) introduce firm asymmetry into Matutes and Regibeau's duopoly two-dimensional Hotelling model, and find that if the dominant firm is sufficiently dominant, then bundling softens competition and both firms benefit. Zhou (2017) adopts a random utility model of Perloff and Salop (1985) to examine bundling in an oligopoly market with n symmetric firms. He finds that when the number of firms is large enough, bundling raises prices and thus benefits firms.

However, neither paper covers the market structure illustrated in the earlier examples: an oligopoly market with a dominant firm and several small firms. Equally important, neither paper studies the hybrid bundling case mentioned in the examples: The dominant firm bundles, while small firms sell separately. In Zhou's setting, since all firms are symmetric, it is natural that he compares only two cases: All firms adopting separate sales and all firms bundling. In HJM, since there are only two firms, when the dominant firm bundles, the small firm effectively bundles as well. As a result, HJM also compare only two cases: both firms selling separately and both firms adopting bundling.

In this paper, we study (pure) bundling under a more general market structure.[2] In particular, we consider an oligopoly market with a dominant firm and several small firms, and we are interested in the following questions. How do firms' different combinations of bundling choices affect their market shares, prices, and profits in the pricing game? Does the dominant firm have a stronger incentive to bundle than small firms? Could the hybrid bundling case,

[2] Mixed bundling, in which a firm offers separate products and a bundle, is not considered.

in which the dominant firm bundles while small firms sell separately, arise as an equilibrium outcome? And how does the equilibrium outcome (in terms of firms' bundling choices) change with the dominance level and the number of firms?

Following Zhou (2017), we adopt a random utility framework. Specifically, we consider an oligopoly market in which both the dominant firm (firm D) and N symmetric small firms produce two products, A and B. Each consumer has a unit demand for product A and product B. Consumers' valuation for a firm's product is randomly drawn from some distribution, and consumers' match values are i.i.d. across firms, consumers, and products. For each product of firm D, each consumer derives an additional utility of $\alpha > 0$. The parameter α captures the dominance level of firm D. We consider three regimes based on firms' bundling decisions: separate sales, in which all firms sell separately; pure bundling, in which all firms bundle; and a hybrid regime, in which the dominant firm bundles and small firms sell separately (this regime is new to the literature, but is commonly observed in practice as noted earlier).[3] We call this model *competing against generalists.*

We also study an alternative but closely related model, which we label *competing against specialists.* Specifically, while firm D still sells two products, each small firm only sells one product, either A or B, and in total there are $2N$ small firms. Since now small firms do not have the option to bundle, we consider only two regimes: separate sales and the hybrid regime, in which firm D bundles (again, this regime is new to the literature). Both models are relevant for real-world applications, depending on whether small firms are multi-product firms.

For each regime of each model, we first characterize the unique quasi-symmetric equilibrium (all small firms charge the same price) in the pricing game.[4] We then compare equilibrium outcomes across regimes in each model. Under each regime, we find that competition among firms can be decomposed into the competition between the dominant firm and small firms and the competition among small firms, with the latter being independent of the dominance level α. This decomposition greatly facilitates comparison across regimes.

In the model of competing against specialists, we find that the dominant firm's bundling hurts itself when the dominance level α is small, but it benefits when α is large. Therefore, separate sales is the equilibrium regime when α is small, and the hybrid regime emerges as an equilibrium when α is large. Intuitively, when firm D bundles, its bundle competes with the best possible "bundle" among all small firms' products. Relative to separate sales, it has three effects: the *demand size effect*, the *marginal consumer effect*, and the *strategic effect.*

The demand size effect can be further decomposed into the *mix and match effect* and the *dispersion effect.* The mix and match effect always reduces firm D's demand, as its bundling

[3] This regime is new in the sense that its equilibrium outcome in the pricing game has not been studied in the literature.

[4] The regime of separate sales leads to the same pricing equilibrium in both models.

removes its products from consumers' choice set of mix and match. The dispersion effect results from the fact that bundling causes the relevant distributions of consumers' match values to be less dispersed and to have thinner tails. When the dominance level α is high, the average position of the consumers that firm D may lose to small firms is in the left tail of firm D's distribution, and in the right tail of small firms' distribution. Since bundling leads to thinner tails, it increases firm D's demand. On the other hand, when α is low, the average position of the "competing" consumers is close to the middle of firms' distributions. Since bundling makes the distributions more peaked in the middle, it reduces firm D's demand.

The marginal consumer effect exhibits a similar pattern. When α is high, the set of marginal consumers (between firm D and small firms) is in the tails of firms' distributions. Thus bundling reduces the set of marginal consumers and softens competition. When α is low, the set of marginal consumers is in the middle of firms' distributions, and thus bundling increases the set of marginal consumers and intensifies competition.

The strategic effect of firm D's bundling tends to make small firms compete less aggressively. Intuitively, under separate sales, a small firm's price decrease only increases its own demand. But when firm D bundles, a small firm's (say in market A) price decrease also benefits small firms in market B, which creates an externality that small firms fail to internalize.

Taken together, when α is small bundling reduces firm D's demand and intensifies competition by increasing the set of marginal consumers, and thus the dominant firm will not bundle. On the other hand, when α is large, bundling increases firm D's demand and softens competition by reducing the set of marginal consumers, and thus the dominant firm will bundle. Our numerical examples show that α could be relatively small for the dominant firm to choose to bundle (roughly firm D's equilibrium market share under separate sales exceeds 60%).

In the model of competing against generalists, the equilibrium regime exhibits the following pattern. When α is small (and the number of firms is not too large), separate sales is the equilibrium regime; when α is intermediate, the hybrid regime in which only firm D bundles is the equilibrium; and when α is sufficiently large, all firms' bundling is the equilibrium regime. Our numerical examples show that while the first two regimes emerge as equilibrium for a wide range of parameter values, the third regime is economically insignificant: All firms' bundling is an equilibrium regime only when the dominant firm is overwhelmingly dominant (roughly, its market share exceeds 90% under separate sales).

The dominant firm's incentive to bundle follows the same pattern as in the model of competing against specialists. To understand small firms' incentive to bundle, consider the case in which α is relatively large and firm D bundles. Relative to the hybrid regime, small firms' bundling causes three effects. First, it reduces small firms' demand, since small firms' bundling prevents consumers from mixing and matching among their products, which amplifies firm D's advantage.[5] Second, it tends to reduce the set of marginal consumers between firm D and small firms, and thus softens competition. Finally, it leads to a strategic effect that causes small firms to price more aggressively. This is because when all firms bundle, a decrease in a small firm's price increases only its own demand, while under the hybrid regime such a decrease also increases the demand of other small firms. Taken together, when α is not too large the mix and match effect and the strategic effect dominate and small firms have no incentive to bundle. On the other hand, when α is sufficiently large, the marginal consumer effect dominates and small firms will bundle.

Since in competitive settings pure bundling can also be interpreted as product incompatibility (in systems markets), our results also shed light on the relationship between market structure and firms' compatibility decisions. Specifically, our results suggest that when the level of market dominance is intermediate, the dominant firm will make its products incompatible with other firms' products, but small firms' products are compatible with each other.

The literature on bundling can be classified into two strands. The first studies bundling in a monopoly context,[6] and the second studies competitive bundling, which is more closely related to our paper. Following Matutes and Regibeau (1988), later work on competitive bundling typically adopts their duopoly two-dimensional Hotelling framework.[7] Also using a spatial framework, Economides (1989) extends Matutes and Regibeau (1988) to an arbitrary number of (symmetric) firms and confirms their result that bundling intensifies competition. In his model, however, firms compete only locally. More recently, Kim and Choi (2015) propose an alternative spatial model in which there are n symmetric firms and firms are not confined to local competition. They find that bundling softens competition if the number of firms is greater than four.

As mentioned earlier, in the literature on competitive bundling, the two papers closely related to ours are HJM and Zhou (2017). Relative to Zhou's n-firm model of symmetric firms, we consider asymmetric firms by introducing a dominant firm. Relative to HJM's model of two asymmetric firms, ours is an n-firm model with firm asymmetry. Qualitatively, the hybrid regime, in which the dominant firm bundles and other small firms sell separately, is not considered by either HJM or Zhou, but is central to our analysis. To summarize, our

[5] Technically, this is because the average of two (i.i.d.) first-order statistics of N i.i.d. random variables first-order stochastically dominates the first-order statistic of N i.i.d. random variables, each being the average of two i.i.d. random variables.

[6] A non-exhaustive list includes Schmalensee (1984) and Fang and Norman (2006) on pure bundling; and Adams and Yellen (1976), McAfee et al. (1989), and Chen and Riordan (2013) on mixed bundling. A complete survey of bundling can be found in Choi (2012).

[7] These papers include Nalebuff (2000) on competitive pure bundling, and Matutes and Regibeau (1992), Thanassoulis (2007), and Armstrong and Vickers (2010) on competitive mixed bundling. Only one does not adopt a spatial model: Anderson and Leruth (1993), who use a logit model to study competitive mixed bundling in duopoly.

paper offers a more complete analysis of the impacts of market structure on firms' incentives to bundle. Further differences between our results and those of HJM and Zhou (2017) will be explained later in the text.

Like HJM, our paper contributes to the literature on the leverage theory of tying (Whinston, 1990): By tying, a monopolist in one market can leverage its monopoly power to another market and thus deter entry or induce exit of rival firms.[8] Our model of competing against specialists generalizes the market structure in the leverage theory, from monopoly in one market to oligopoly with a dominant firm. Moreover, we show that when the level of market dominance large enough, it is in its own (static) interest for the dominant firm to bundle or practice tying.

The rest of the paper is organized as follows. Section 2 sets up the model. In Section 3 we analyze the model of competing against specialists, and in Section 4 we study the model of competing against generalists. Section 5 offers concluding remarks. All proofs can be found in the Appendix.

2 The Model

We present two models. The first one is a multi-product dominant firm competing against N generalists, and the second one is a multi-product dominant firm competing against $2N$ specialists. Each of them is of independent interests, depending on applications.

2.1 One dominant firm and N generalists

There are $N+1$ ($N \geq 2$) multi-product firms, each producing two products/components, A and B.[9] Among the firms, firm D is a dominant firm, while the remaining N firms are small and symmetric. Denote ij as product j produced by firm i, for $i = D, 1, ..., N$ and $j = A, B$. The marginal cost of each product ij is normalized to 0, thus prices can be considered as markups. The measure of consumer is 1. Each consumer has a unit demand for each product j.

We adopt the random utility model of Perloff and Salop (1985) and Zhou (2017) to model product differentiation. Specifically, a consumer's gross (match) utility from product ij is X_{ij}. We assume that X_{ij} is i.i.d. across consumers, and for any given consumer it is i.i.d across firms. Moreover, for any firm i, X_{iA} and X_{iB} are i.i.d. as well. Let the CDF of the common distribution of all X_{ij} be $F(x)$, and $f(x)$ be its PDF, with a bounded support $[\underline{x}, \overline{x}]$. We assume $f(x)$ is continuously differentiable, logconcave, and $f(x) > 0$ for any $x \in [\underline{x}, \overline{x}]$.

For each of the dominant firm's products, a consumer derives an additional utility $\alpha > 0$

[8] Also see Choi and Stefanadis (2001), Carlton and Waldman (2002), and Nalebuff (2004). Other papers studying bundling of a multiproduct firm that competes against single-product rivals include Carbajo et al. (1990), Chen (1997), and Denicolo (2000).

[9] The case $N = 1$ is essentially the same as the model in HJM.

(sometimes we also consider the limiting case of $\alpha = 0$). This α can be interpreted as the quality advantage of firm D's products over other firms' products,[10] which measures the degree of market dominance of the dominant firm. Note that firm D's dominance level in market A and that in market B are the same, which means market A and market B are symmetric. This assumption is made for simplicity.[11] If a consumer buys two products with match utilities (x_A, x_B) from small firms and pays total payment T, then his net utility is $x_A + x_B - T$. If a consumer buys one product (two products) from firm D, then α (2α) is added to his net utility. Finally, we assume full market coverage, which means that each consumer will always buy exactly one good A and one good B.[12]

The game is played in two stages. In the first stage, firms simultaneously choose whether to bundle their own products, and then their decisions become public. In the second stage all firms simultaneously set prices. Finally, consumers purchase their products after observing all the prices and match utilities. We only consider pure bundling for each individual firm: a firm either offers a pure bundle or sells two products separately. Mixed bundling (a firm offers bundle and separate products at the same time) is not considered in this paper.

We will consider quasi-symmetric equilibria in which small firms adopt symmetric strategies in equilibrium. Notice that, if all small firms choose to bundle, then essentially the dominant firm automatically bundles as well. Consequently there are three possible regimes based on the first stage bundling decisions: separate sales, pure bundling, and only firm D bundles. Under separate sales, each firm sells two products separately, while under pure bundling each firm sells its two products only as a bundle. In the last (hybrid) regime, firm D sells its two products as a bundle, while each small firm sells its two products separately.

2.2 One dominant firm and $2N$ specialists

In the case of competing against specialists, firm D still produces two products, but each small firm only produces one product. In particular, there are $2N$ small firms, with N small firms producing product A and the other N firms producing product B. We thus can index a small firm as ij by the product ij it produces. All the other assumptions are the same as in the other model. Note that all small firms are again symmetric, and market A and market B are symmetric as well. In this setting, since firm D is the only multi-product firm, only firm D has an option to bundle. We thus only consider two regimes: separate sales and firm D bundles.

[10] It also could be interpreted as firm D's advantage in the marginal cost of production.

[11] In the conclusion, we discuss how our main results will change when this assumption is relaxed.

[12] This full coverage assumption is typically made in the literature of competitve bundling. For instance, both HJM and Zhou (2017) made this assumption.

3 Competing against Specialists

We first study the model of competing against specialists, as it is simpler than the other model. We will first characterize the equilibrium outcomes under the two regimes, and then compare them.

3.1 Separate sales

Under separate sales all firms sell separately, thus consumers are free to mix and match. As a result, market A and market B are independent and symmetric. We can thus analyze them separately. Consider the market for product j. We focus on quasi-symmetric equilibria, in which all small firms charge the same price p and firm D charges p_D. Denote $q_D(p_D, p)$ as firm D's demand, and $q(p', p, p_D)$ as a small firm ij's demand when it charges p', firm D sets p_D, and all other small firms charge p. Firms' demand can be expressed as follows:

$$
\begin{aligned}
q_D(p_D, p) &= \Pr[X_{Dj} + \alpha - p_D > \max_{k \neq D}\{X_{kj}\} - p], \\
q(p', p, p_D) &= \Pr[X_{ij} - p' > \max_{k \neq D, i}\{X_{kj}\} - p \text{ and } X_{ij} - p' > X_{Dj} + \alpha - p_D].
\end{aligned}
$$

Lemma 1 *Under separate sales, (i) $q_D(p_D, p)$ is continuous, differentiable, and logconcave in p_D; (ii) $q(p', p, p_D)$ is continuous, differentiable, and logconcave in p'.*

The results in Lemma 1 are standard. Essentially, the logconcavity of demand in its own price results from the facts that X_{ij}'s have logconcave densities and they are independent from each other. The logconcavity of the demand further implies that each firm's best response is unique, and the first-order conditions are sufficient.

To characterize quasi-symmetric equilibria, let

$$
\Delta = \alpha - p_D + p, \tag{1}
$$

be firm D's net advantage. It will be verified later that in equilibrium $\Delta > 0$. Let F_N be the CDF of the first-order (or the largest) statistic of N i.i.d. random variables, each distributed according to F, and $f_N = NF^{N-1}f$ be its PDF. F_{N-1} and f_{N-1} are defined accordingly. More explicitly,

$$
\begin{aligned}
q_D(p_D, p) &= \prod_{i=1...N} \Pr[X_{Dj} > X_{ij} - \Delta] = 1 - \int_{\underline{x}+\Delta}^{\overline{x}} F(x - \Delta) f_N(x) dx; \qquad (2) \\
q(p', p, p_D) &= \prod_{k \neq i, D} \Pr[X_{ij} > X_{kj} + p' - p] \cdot \Pr[X_{ij} > X_{Dj} + \Delta + p' - p] \\
&= \int_{\underline{x}+p'-p}^{\overline{x}} [F(x + p - p')]^{N-1} f(x) dx \cdot \int_{\underline{x}+\Delta+p'-p}^{\overline{x}} F(x + p - p' - \Delta) f_N(x) dx.
\end{aligned}
$$

In the last expression, the two probabilities can be separated because the second probability is conditional on the first event: product ij is the best product j among all small firms' products. When $p' = p$, a small firm's demand $q(p, p_D)$ becomes

$$q(p, p_D) = \frac{1}{N} \int_{\underline{x}+\Delta}^{\overline{x}} F(x-\Delta) f_N(x) dx = \frac{1}{N}[1 - q_D(p_D, p)]. \tag{3}$$

Firm D chooses p_D to maximize $\pi_D = p_D q_D(p, p_D)$, and a small firm chooses p' to maximize $\pi = p' q(p', p, p_D)$. The first-order conditions (after imposing symmetry $p' = p$) yield

$$p_D = \frac{1 - \int_{\underline{x}+\Delta}^{\overline{x}} F(x-\Delta) f_N(x) dx}{\int_{\underline{x}+\Delta}^{\overline{x}} f(x-\Delta) f_N(x) dx}, \tag{4}$$

$$p = \frac{\frac{1}{N} \int_{\underline{x}+\Delta}^{\overline{x}} F(x-\Delta) f_N(x) dx}{\int_{\underline{x}}^{\overline{x}} f_{N-1}(x) f(x) dx \cdot \int_{\underline{x}+\Delta}^{\overline{x}} F(x-\Delta) f_N(x) dx + \frac{1}{N} \int_{\underline{x}+\Delta}^{\overline{x}} f(x-\Delta) f_N(x) dx}, \tag{5}$$

$$\Leftrightarrow \quad \frac{1}{p} = k_S + \frac{\int_{\underline{x}+\Delta}^{\overline{x}} f(x-\Delta) f_N(x) dx}{\int_{\underline{x}+\Delta}^{\overline{x}} F(x-\Delta) f_N(x) dx}, \tag{6}$$

where $k_S = N \int_{\underline{x}}^{\overline{x}} f_{N-1}(x) f(x) dx$, which is independent of Δ. A quasi-symmetric equilibrium is characterized by equations (1), (4), and (5).

The denominators in (4) and (5) measure a firm's set of marginal consumers, who are indifferent between its product and the best product among all other firms. Specifically, for a small firm it has two terms. The first term is the set of marginal consumers when the best alternative product is another small firm's product, while the second term is the set of marginal consumers when the best alternative product is firm D's. Conceptually, we can decompose the competition among firms into two steps. In the first step, firm D's product competes with the best product among small firms' products. This determines firm D's demand and small firms' aggregate demand (the term $\int_{\underline{x}+\Delta}^{\overline{x}} F(x-\Delta) f_N(x) dx$). In the second step, small firms compete among themselves, which determines the allocation of small firms' aggregate demand among small firms (equal allocation in quasi-symmetric equilibrium). According to this decomposition, Δ only affects the first step of competition, which is reflected in k_S being independent of Δ in equation (6).

Proposition 1 *Under separate sales, there is a unique quasi-symmetric equilibrium in the pricing game. In the unique quasi-symmetric equilibrium, (i) $p_D > p$, $q_D > q$, and $0 < \Delta < \alpha$; (ii) as α increases, Δ, p_D, and q_D all increase, while q and p decrease.*

Quint (2014) establishes the existence and uniqueness of equilibrium in a more general setting. Our proof is simpler and more transparent, adapted to quasi-symmetric equilibrium.[13]

[13] Following Quint's (2014) uniquenss result, in our model the quasi-symmetric equilibrim is the unique equilibrium, or equilibrium that is not quasi-symmetric does not exist. Relatedly, Caplin and Nalebuff (1991) show

Similar proofs will be applied to the bundling cases, which are not covered by Quint (2014). Parts (i) and (ii) in Proposition 1 are intuitive. Since price and demand are complements (profit is the product of the two), firm D will "spend" its gross advantage α relatively equally on boosting demand (a positive Δ) and charging a higher price. As α increases, both the net advantage Δ and the price difference between p_D and p increase in equilibrium.

3.2 Firm D bundles

When firm D bundles, it creates a link between market A and market B. Thus we have to consider two markets jointly. Note that consumers are free to mix and match among all small firms' products. Thus firm D's bundle competes with the best possible combination of products among all small firms' products. Again we are interested in quasi-symmetric equilibria in which all small firms charge the same price p. Let P_D be half of firm D's bundle price. Note that all small firms' positions are symmetric.

In order to derive firms' demand, let $\overline{X}_{Nj} \sim F_N$ be the best match utility among all small firms' product j. Define $Z_N = (\overline{X}_{NA} + \overline{X}_{NB})/2$, which indicates the average match utility of the "best bundle" among all small firms' products (the support of Z_N is still $[\underline{x}, \overline{x}]$). Denote G_N and g_N as the CDF and PDF of Z_N, respectively. Similarly, we define $Z_D = (X_{DA} + X_{DB})/2$ as the average match utility of firm D's bundle, and G and g as the CDF and PDF of Z_D, respectively. Again, we define $\Delta = \alpha - P_D + p$ as firm D's net advantage per product.

Firm D's demand $q_D(P_D, p)$ is given by

$$q_D(P_D, p) = \Pr[Z_D > Z_N - \Delta] = 1 - \int_{\underline{x}+\Delta}^{\overline{x}} G(z-\Delta) g_N(z) dz. \tag{7}$$

Now consider a small firm ij. Given that all other small firms charge p and firm D charges p_D, if it charges p', then its demand $q(p', p, P_D)$ is given by

$$\begin{aligned} q(p', p, P_D) \quad &= \\ \Pr[X_{ij} \quad &> \quad \max_{k \neq D,i}\{X_{kj}\} - p + p' \text{ and } (X_{ij} - p') + \max_{k \neq D}\{X_{k(-j)}\} - p > 2Z_D + 2\alpha - 2P_D], \end{aligned}$$

which can be explicitly written as

$$q(p', p, P_D) = \int_{\underline{x}}^{\overline{x}} [F(x + p - p')]^{N-1} f(x) dx \int_{\underline{x}+\Delta}^{\overline{x}} G(z - \Delta + p/2 - p'/2) g_N(z) dz. \tag{8}$$

When $p' = p$, a small firm's demand $q(p, P_D)$ becomes

$$q(p, P_D) = \frac{1}{N} \int_{\underline{x}+\Delta}^{\overline{x}} G(z-\Delta) g_N(z) dz = \frac{1}{N}[1 - q_D(p, P_D)]. \tag{9}$$

that there is no asymmetric equilibrium in the logit model.

To understand the expressions, note that for product ij to be sold two conditions have to be met. First, it is the best product j among all small firms' products. Second, product ij plus the best product $-j$ among all small firms' products beat firm D's bundle. In the expression of (8), the two probabilities can be separated because the second probability is conditional on the first event: product ij is the best product j among all small firms' products.

Lemma 2 *When firm D bundles, (i) $q_D(P_D, p)$ is continuous, differentiable, and logconcave in P_D; (ii) $q(p', p, P_D)$ is continuous, differentiable, and logconcave in p'.*

That each demand is logconave in its own price is because f is logconcave, the first-order statistic of i.i.d. random variables inherits logconcavity, and convolution of independent random variables also preserves logconcavity. Each demand being logconcave in its own price further implies that each firm's best response is unique, and the first-order conditions are sufficient.

The first-order condition for firm D yields:

$$P_D = \frac{1 - \int_{\underline{x}+\Delta}^{\overline{x}} G(z-\Delta) g_N(z)dz}{\int_{\underline{x}+\Delta}^{\overline{x}} g(z-\Delta) g_N(z)dz}. \tag{10}$$

For a small firm, taking derivative of $p'q(p', p, P_D)$ with respect to p' and then imposing symmetry, we get

$$p = \frac{\frac{1}{N}\int_{\underline{x}+\Delta}^{\overline{x}} G(z-\Delta) g_N(z)dz}{\int_{\underline{x}}^{\overline{x}} f_{N-1}(x)f(x)dx \int_{\underline{x}+\Delta}^{\overline{x}} G(z-\Delta) g_N(z)dz + \frac{1}{2N}\int_{\underline{x}+\Delta}^{\overline{x}} g(z-\Delta) g_N(z)dz}, \tag{11}$$

$$\Leftrightarrow \quad \frac{1}{p} = k_S + \frac{1}{2}\frac{\int_{\underline{x}+\Delta}^{\overline{x}} g(z-\Delta) g_N(z)dz}{\int_{\underline{x}+\Delta}^{\overline{x}} G(z-\Delta) g_N(z)dz}. \tag{12}$$

A quasi-symmetric equilibrium is characterized by (10), (11) and $\Delta = \alpha - P_D + p$.

Again, conceptually we can decompose competition among firms into two steps. In the first step, firm D's bundle competes with the best possible "bundle" among small firms' products, which determines firm D's demand and small firms' aggregate demand (the term $\int_{\underline{x}+\Delta}^{\overline{x}} G(z - \Delta)g_N(z)dz$). In the second step, small firms' competition among themselves determines the allocation of small firms' aggregate demand. Note that firm D's bundling does not affect small firms' competition among themselves: the term k_S is the same in both (6) and (12).

Proposition 2 *When firm D bundles, there is a unique quasi-symmetric equilibrium in the pricing game. Moreover, in equilibrium (i) as α increases, Δ, p_D, and q_D all increase, while q and p decrease; (ii) there exists an $\widehat{\alpha} \in (0, \overline{x} - \underline{x})$ such that if $\alpha \geq \widehat{\alpha}$, then $\Delta < \alpha$ and $P_D > p$.*

Proposition 2 is almost identical to Proposition 1, except for one notable difference. When firm D bundles and if the dominance level α is low, P_D could be smaller than p. This is because for small α, firm D's bundling will reduce its demand, the intuition of which will be explained shortly.

3.3 Comparison

In this subsection we compare the equilibrium outcomes under two regimes. Denote SS as the regime of separate sales, and DB as the regime under which firm D bundles. Relative to SS, DB changes the relevant distributions of consumers' match values. Specifically, for the competition between firm D and small firms, the relevant distributions change from F and F_N to G and G_N, respectively. The key observation is that G is a mean-preserving contraction of F, and G_N is also a mean-preserving contraction of F_N. That is, G has thinner tails than F, and G_N also has thinner tails than F_N. This is because the average of two i.i.d. match values is less dispersed than the original match value. Figure 1 illustrates the pattern when $N = 2$ and F is uniform on $[0, 1]$, which is our leading example.

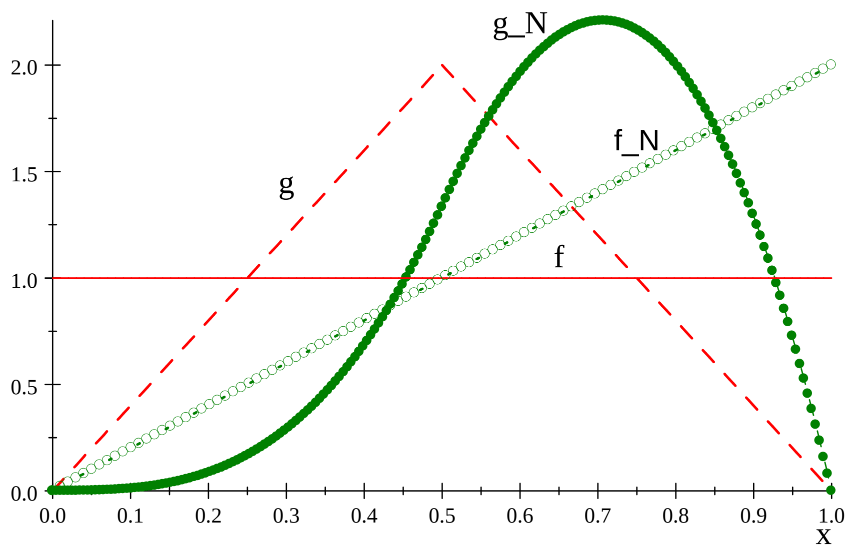

Figure 1: Density functions of F, G, F_N and G_N.

The comparison of the left tail between f and g and the comparison of the right tail between f_N and g_N turn out to be important. The next lemma formally states the pattern.

Lemma 3 *There exist x_1 and x_2 such that, for $x \in [\underline{x}, x_1)$ $g(x) < f(x)$, and for $x \in (x_2, \overline{x}]$ $g_N(x) < f_N(x)$.*

Lemma 3 always holds because $f(\underline{x}) > 0$ and $f_N(\overline{x}) > 0$ (since $f(\overline{x}) > 0$), and $g(\underline{x}) = 0$ and $g_N(\overline{x}) = 0$ (convolution leads to 0 density at the boundaries). In the example of uniform distribution, Figure 1 indicates that $x_1 = 0.25$ and $x_2 = 0.85$. The next proposition shows that firm D's incentive to bundle depends crucially on the dominance level α.

Proposition 3 *In the model of competing with specialists the following results hold. (i) If α is big enough such that the equilibrium $\Delta^{SS} \to \overline{x} - \underline{x}$, then $\pi_D^{DB} > \pi_D^{SS}$ (DB is the equilibrium regime), and $P_D^{DB} > p_D^{SS}$. (ii) If α is big enough such that the equilibrium $\Delta^{SS} \geq \max\{\overline{x} - x_1, x_2 - \underline{x}\}$, where x_1 and x_2 are defined in Lemma 3, and the following condition holds*

$$\frac{\int_{\underline{x}+\Delta}^{\overline{x}} f(z-\Delta) f_N(z) dz}{\int_{\underline{x}+\Delta}^{\overline{x}} F(z-\Delta) f_N(z) dz} \geq \frac{1}{2} \frac{\int_{\underline{x}+\Delta}^{\overline{x}} g(z-\Delta) g_N(z) dz}{\int_{\underline{x}+\Delta}^{\overline{x}} G(z-\Delta) g_N(z) dz} \tag{13}$$

for $\Delta = \Delta^{SS}$, then $P_D^{DB} > p_D^{SS}$, $p^{DB} > p^{SS}$, and $\pi_D^{DB} > \pi_D^{SS}$ (firm D will bundle). (iii) If $\alpha \to 0$ and

$$\int_{\underline{x}}^{\overline{x}} g(z) g_N(z) dz > \int_{\underline{x}}^{\overline{x}} f(z) F_N(z) dz, \tag{14}$$

and in equilibrium $p^{DB} \leq p^{SS}$, then $\pi_D^{DB} < \pi_D^{SS}$ (firm D will not bundle).

Parts (i) and (ii) of Proposition 3 show that firm D will bundle if it is sufficiently dominant. While part (i) is a limiting result, part (ii) shows that firm D will bundle even away from the limit. Condition (13) in part (ii) ensures that at a high dominance level α, small firms price less aggressively under DB than under SS. In the uniform example with $N = 2$, this condition is satisfied whenever $\Delta \geq 1/2$. Part (iii) indicates that firm D will not bundle if α is sufficiently small. Condition (14) and $p^{DB} \leq p^{SS}$ ensure that DB intensifies competition between firm D and small firms. Again, these two conditions are satisfied in the uniform example with $N = 2$.

Before discussing the intuition behind Proposition 3, we first present an example to illustrate that for DB to be the equilibrium regime α could be relatively small. Under the uniform example with $N = 2$, Figure 2 illustrates firms' equilibrium prices, market shares, and profits (in panel 1, 2 and 3, respectively), as α changes.

Several observations are in order. First, when α is larger than 0.6 (firm D's equilibrium market share under SS at $\alpha = 0.6$ is about 58%) firm D's profit under DB is higher than its profit under SS, while when α is smaller than 0.6 firm D's profit is higher under SS. This shows that firm D will bundle even when the dominance level α is fairly small. Second, relative to SS, firm D's equilibrium price is lower when α is relatively small and higher when α is relatively large under DB (the cutoff α is about 1.35), and the difference in prices gets larger as α increases. Small firms' equilibrium price exhibits a similar pattern. Finally, when α is not too large firm D' demand is higher under DB than under SS but the difference in demands

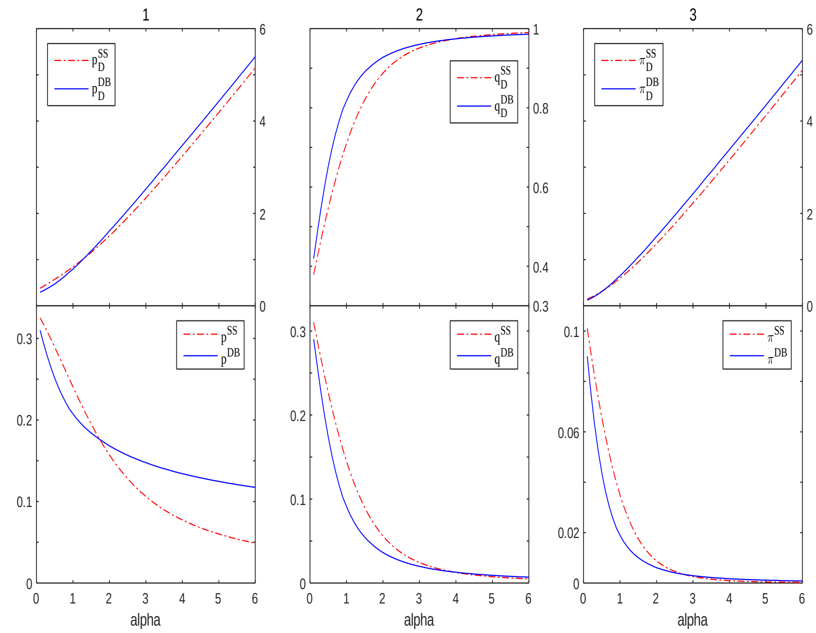

Figure 2: Equilibrium prices, market shares and profits as α changes

vanishes as α increases; when α is sufficiently large firm D's demand is lower under DB than under SS.

To understand the intuition behind these results, we decompose the impacts of firm D's bundling into the following three effects: the *demand size effect,* the *marginal consumer effect,* and the *strategic effect*. We proceed to discuss these effects in detail.

The demand size effect. This can be further decomposed into two effects. We call the first one the *dispersion effect*, as DB makes the relevant distributions of consumers' match values less dispersed. The second effect is the *mix and match effect*, which refers to the fact that DB excludes firm D's products from consumers' choice set of mix and match. This effect is always negative, since losing the flexibility of mix and match tends to reduce firm D's demand.

However, the dispersion effect is more complicated, as it depends on the dominance level α and the number of small firms N. More precisely, under SS the set of consumers that firm D loses to small firms is $\int_{\underline{x}+\Delta}^{\overline{x}} F(x-\Delta) f_N(x)dx$. We call this the set of competing consumers, which becomes $\int_{\underline{x}+\Delta}^{\overline{x}} G(x-\Delta) g_N(x)dx$ under DB. Note that for any competing consumer, his match value from firm D lies between $\underline{x}$ and $\overline{x}-\Delta$ and his best match value from small firms lies between $\underline{x}+\Delta$ and $\overline{x}$. As α increases and thus Δ increases, the average position of competing consumers is pushed to the left in firm D's distribution, while their average position is pushed to the right in small firms' distribution. Recall that DB leads to thinner tails for the relevant distributions (G (G_N) has thinner tails relative to F (F_N)). This means that if α is big enough then DB reduces the set of competing consumers, which increases firm D's demand. Overall, when α is small the mix and match effect dominates and DB reduces firm D's demand,[14] while when α is large the dispersion effect dominates and DB increases firm D's demand.

The demand size effect is illustrated in Figure 3 (the uniform example with $N = 2$), in which one curve is the equilibrium q_D^{SS} as a function of α, and the other curve is q_D^{DB} as a function of α but using the equilibrium Δ^{SS} under separate sales for each α (in order to remove the effect of different prices). In the figure, we see that DB increases (decreases) firm D's demand when α is higher (lower) than 1.15.

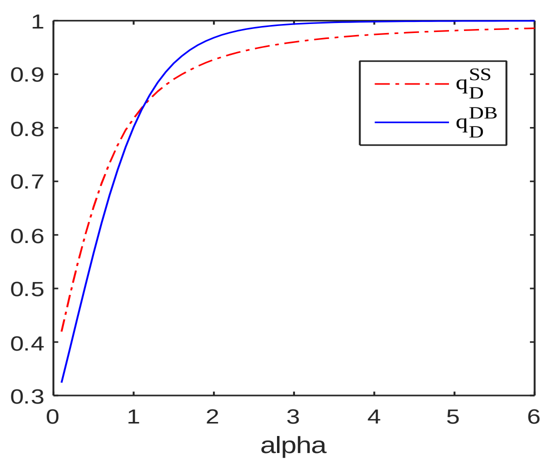

Figure 3: Illustration of the demand size effect

We want to point out that the demand size effect in our model is subtler than the demand size effect in HJM. Since HJM only has two firms in each market, firm D's bundling effectively means that the products of small firms bundle as well. As a result, in their setting the demand size effect is always positive for any $\alpha > 0$. In our setting, having more than two firms in each market leads to two differences. First, under DB consumers still can mix and match among small firms' products. Second, when α is very small, having more than three firms means that DB is very likely to reduce firm D's demand even without the mix and match effect.[15] These differences make the demand size effect subtler in our model.

The marginal consumer effect. DB changes firms' set of marginal consumers, which affects their incentives to set prices. Recall the decomposition of competition mentioned earlier: DB does not affect the competition (the set of marginal consumers) among small firms. Thus we

[14] When α is almost 0, we can formally show that DB reduces firm D's demand. See Section 4 for details.

[15] With $N \geq 2$ and α being small, since firm D's products compete with the best products among small firms, the average position of competing consumers lies in the right tail of firm D's distribution. Thus DB increases the set of competing consumers and reduces firm D's demand.

focus on the set of marginal consumers between firm D and small firms. Specifically, under SS the set of marginal consumers between firm D and small firms is $\int_{\underline{x}+\Delta}^{\overline{x}} f(x-\Delta)f_N(x)dx$. When α and thus Δ is large, the set of marginal consumers is on average in the left tail of firm D's match value distribution and in the right tail of small firms' match value distribution. Thus, by making the tails of relevant distributions thinner, DB reduces the set of marginal consumers. As a result, competition between firms is softened. On the other hand, when α is very small, DB tends to increase the set of marginal consumers as long as N is not too large.[16] Intuitively, when α is close to 0 and N is not too large the average position of marginal consumers is close to the middle of distribution. Thus, by making consumers' match value distribution less dispersed, DB increases the set of marginal consumers, which tends to intensify competition among firms.

The strategic effect. DB directly changes small firms' incentives to set prices. Comparing the two pricing equations (6) and (11), we see that the coefficient before the integral ratio is 1/2 under DB, while under SS it is 1. Other things being equal, this effect tends to increase small firms' price under DB, which benefits firm D. The underlying reason for this effect is that, under DB a small firm's product iA plus the best product of all small firms in market B compete with firm D's bundle. As a result, relative to SS, under DB a one unit reduction of p_{iA} is only half effective in taking consumers away from firm D.[17]

Taken together, when α is small DB reduces firm D's demand and increases the set of marginal consumers, both of which hurting firm D. Thus firm D has no incentive to bundle.[18] On the other hand, when α is big DB increases firm D's demand and reduces the set of marginal consumers, both of which benefiting firm D. Moreover, the strategic effect also helps soften competition. As a result, firm D will bundle under a high dominance level α.

Figure 4 illustrates the equilibrium regime as the number of firms and α change. The left panel is the uniform example, while in the right panel F is a truncated normal distribution on $[-1, 1]$ with $\sigma = 1$.[19] Consistent with earlier discussion, under both distributions and for each N, firm D sells separately when α is small and bundles when α is large.

Another observation held under both distributions is that as the number of small firms N increases, the cutoff α at which the equilibrium regime switches increases. This indicates that firm D's incentive to bundle decreases in the number of small firms. The underlying reason for this pattern is that the mix and match effect becomes stronger as N increases. Recall

[16] Zhou (2017) showed this result for $\alpha = 0$ and $N = 1$ (duopoly).

[17] Put it another way, under DB a reduction in the price of a small firm iA benefits all small firms in market B, but firm iA fails to internalize these benefits, which reduces its incentive to price aggressively.

[18] Actually, when α is very close to 0, under DB firm D gets a lower market share, charges a lower price, and earns a lower profit per product, than small firms.

[19] The upper bound of α is set at the level where the equilibrium market share of a small firm under SS diminishes to 0.05%.

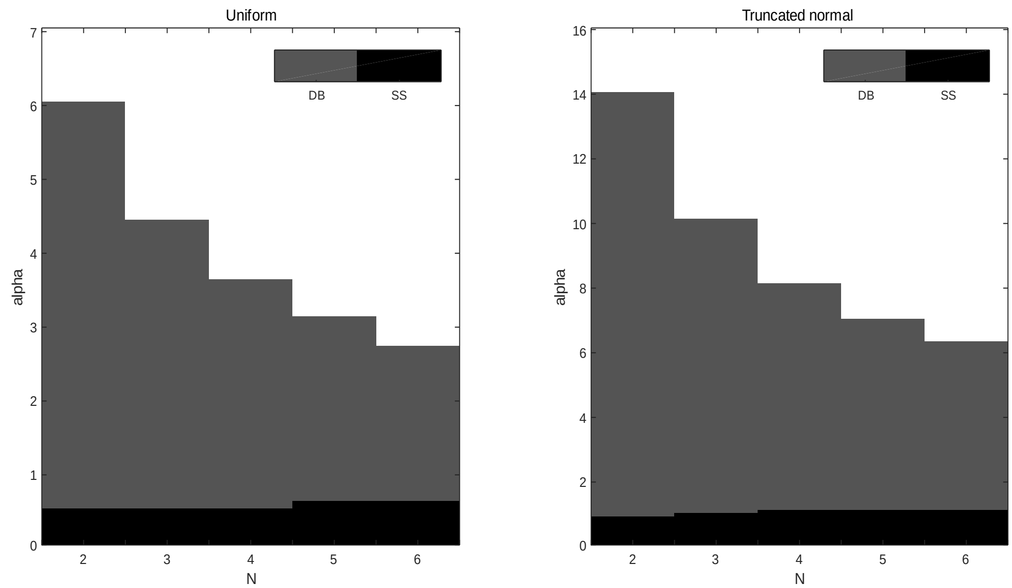

Figure 4: The pattern of equilibrium regimes in the model of competiting with specialists.

that under DB firm D's bundle competes with the best possible "bundle" among all small firms' products. As the number of small firms increases, the competitive advantage of the best possible bundle increases, or firm D's loss from being excluded from consumers' choice set of mix and match is magnified, which reduces firm D's incentive to bundle.

4 Competing against Generalists

Now we come back to the setting where each small firm produces two products (A and B), and there are N small firms in total. Under this setting, recall that there are three regimes: separate sales, only firm D bundles, and pure bundling in which all firms bundle. Again, we will study the equilibrium outcomes under each regime and then compare them.

4.1 Separate sales

Under the regime of separate sales, markets A and B are independent. Therefore, this case is the same as separate sales when firm D competes against specialists: there is a unique quasi-symmetric equilibrium in the pricing game, and it is characterized by equations (1), (4), and (5).

4.2 Pure bundling

Under the regime of pure bundling, each firm bundles its own products A and B. Thus, each consumer compares $N+1$ bundles and then buys one bundle. Let $Z_i = (X_{iA} + X_{iB})/2$ be a consumer's average match utility from firm i's bundle. Using earlier notations, $Z_i \sim G$. Let G_{BN} be the CDF of the first-order statistic of N i.i.d. random variables, each distributed according to G, and $g_{BN} = NG^{N-1}g$ be its PDF. Notice the difference between G_{BN} and G_N defined earlier. While G_{BN} defines the highest match utility among all N small firms' bundles, G_N captures the highest utility of the "best bundle" among all N^2 possible bundles when consumers are allowed to mix and match among small firms' products.

Again, we are interested in quasi-symmetric equilibria in which all small firms charge the same bundle price. Denote P_D and P as the per-product bundle prices (price of the whole bundle divided by 2) for firm D and small firms, respectively. And again let $\Delta = \alpha + P - P_D$ be firm D's net advantage per-product. By Bagnoli and Bergstrom (2005), the logconcavity of f implies the logconcavity of g and g_{BN}. Therefore, the results of Proposition 1 also applies to the pure bundling regime, which yields the following corollary.

Corollary 1 *Under pure bundling, there is a unique pricing equilibrium, which is quasi-symmetric. Moreover, (i) $P_D > P$, $q_D > q$, and $0 < \Delta < \alpha$; (ii) as α increases, Δ, P_D, and q_D all increase, while q and P decrease.*

The characterization of the quasi-symmetric equilibrium is very similar to the one under separate sales. The pricing equations are listed below.

$$P_D = \frac{1 - \int_{\underline{x}+\Delta}^{\overline{x}} G(x-\Delta) g_{BN}(x)dx}{\int_{\underline{x}+\Delta}^{\overline{x}} g(x-\Delta) g_{BN}(x)dx}, \tag{15}$$

$$P = \frac{\frac{1}{N}\int_{\underline{x}+\Delta}^{\overline{x}} G(x-\Delta) g_{BN}(x)dx}{\int_{\underline{x}}^{\overline{x}} g_{B(N-1)}(x)g(x)dx \int_{\underline{x}+\Delta}^{\overline{x}} G(x-\Delta) g_{BN}(x)dx + \frac{1}{N}\int_{\underline{x}+\Delta}^{\overline{x}} g(x-\Delta) g_{BN}(x)dx}, \tag{16}$$

$$\Leftrightarrow \quad \frac{1}{P} = k_B + \frac{\int_{\underline{x}+\Delta}^{\overline{x}} g(z-\Delta) g_{BN}(z)dz}{\int_{\underline{x}+\Delta}^{\overline{x}} G(z-\Delta) g_{BN}(z)dz}, \tag{17}$$

where $g_{B(N-1)} = (N-1)G^{N-2}g$, and $k_B = N\int_{\underline{x}}^{\overline{x}} g_{B(N-1)}(x)g(x)dx$, which is independent of Δ.

4.3 Only firm D bundles

Under the regime that only firm D bundles, again we are interested in quasi-symmetric equilibria in which all small firms charge the same price p for each product. Note that this case is very similar to regime DB in the model of competing against specialists. Thus we adopt the same notations. Indeed, firm D's demand function and best response are exactly the same.

$$
\begin{aligned}
q_D(p, P_D) &= \Pr[Z_D > Z_N - \Delta] = 1 - \int_{\underline{x}+\Delta}^{\overline{x}} G(z-\Delta) g_N(z) dz, \\
P_D &= \frac{1 - \int_{\underline{x}+\Delta}^{\overline{x}} G(z-\Delta) g_N(z) dz}{\int_{\underline{x}+\Delta}^{\overline{x}} g(z-\Delta) g_N(z) dz}.
\end{aligned} \tag{18}
$$

Now we compute a small firm's demand, which is different from the one under regime DB when firm D competes against specialists. We restrict our attention to symmetric strategies: each small firm charges the same price for its product A and product B. With symmetric prices, markets A and B are symmetric. Suppose a small firm i charges p', all other small firms charge p for each product, and firm D charges $2P_D$ for its bundle. Then, firm i's demand per-product (say product A) is

$$
q(p', p, P_D) = \int_{\underline{x}}^{\overline{x}} [F(x + p - p')]^{N-1} f(x) dx \cdot [Q_1 + Q_2],
$$

where

$$
\begin{aligned}
Q_1 &= \int_{\underline{x}}^{\overline{x}} [F(x + p - p')]^{N-1} f(x) dx \int_{\underline{x}+\Delta}^{\overline{x}} G(z - \Delta + p - p') g_N(z) dz, \\
Q_2 &= \int_{\underline{x}}^{\overline{x}} [1 - [F(x + p - p')]^{N-1}] f(x) dx \int_{\underline{x}+\Delta}^{\overline{x}} G(z - \Delta + p/2 - p'/2) g_N(z) dz.
\end{aligned}
$$

To understand the above expressions, Q_1 is firm i's demand for product iA when product iA is the best among all small firms' product A, product iB is the best among all small firms' product B, and the bundle iA and iB is better than firm D's bundle. Similarly, Q_2 is firm i's demand for product iA when product iA is the best among all small firms' product A, product iB is not the best among all small firms' product B, and the bundle iA plus the best product among all other small firms' product B is better than firm D's bundle.

Lemma 4 *Under the regime that only firm D bundles, (i) q_D is logconcave in P_D, and π_D is quasiconcave in P_D; (ii) both Q_1 and Q_2 are logconcave in p'.*

Lemma 4 implies that firm D has a unique best response for any given p, and the first-order condition is sufficient. However, we are not able to show that $Q_1 + Q_2$ is logconcave in p'. The underlying reason is that we cannot combine Q_1 and Q_2 into a single term.[20]

[20] The sum of two logconave functions may not be logconcave. More precisely, $\int_{\underline{x}+\Delta}^{\overline{x}} G(z - \Delta + p - p') g_N(z) dz$ (iA and iB beat D's bundle) is different from $\int_{\underline{x}+\Delta}^{\overline{x}} G(z - \Delta + p/2 - p'/2) g_N(z) dz$ (iA and jB beat D's bunble).

When $p' = p$, a small firm's per-product demand becomes

$$q(p, P_D) = \frac{1}{N}\int_{\underline{x}+\Delta}^{\overline{x}} G(z-\Delta)g_N(z)dz = \frac{1}{N}(1-q_D).$$

Note that this is the same as $q(p, P_D)$ under regime DB in the model of competing against specialists. The first-order condition for small firms can be derived as:

$$p = \frac{\frac{1}{N}\int_{\underline{x}+\Delta}^{\overline{x}} G(z-\Delta)g_N(z)dz}{\int_{\underline{x}}^{\overline{x}} f_{N-1}(x)f(x)dx \int_{\underline{x}+\Delta}^{\overline{x}} G(z-\Delta)g_N(z)dz + \frac{N+1}{2N^2}\int_{\underline{x}+\Delta}^{\overline{x}} g(z-\Delta)g_N(z)dz}, \tag{19}$$

$$\Leftrightarrow \quad \frac{1}{p} = k_S + \frac{N+1}{2N}\frac{\int_{\underline{x}+\Delta}^{\overline{x}} g(z-\Delta)g_N(z)dz}{\int_{\underline{x}+\Delta}^{\overline{x}} G(z-\Delta)g_N(z)dz}. \tag{20}$$

A quasi-symmetric equilibrium is characterized by a three-equation system: (18), (19), and $\Delta = \alpha + p - P_D$. The next proposition shows that the three-equation system has a unique solution.

Proposition 4 *In the model of competing against generalists, under the regime that only firm D bundles, the three equation system characterizing the quasi-symmetric equilibrium has a unique solution. Moreover, in equilibrium (i) as α increases, Δ, P_D, and q_D all increase, while q and p decrease; (ii) there exists an $\widehat{\alpha} \in (0, \overline{x} - \underline{x})$ such that, if $\alpha \geq \widehat{\alpha}$, then $\Delta < \alpha$ and $P_D > p$.*

Since the three-equation system is necessary (may not be sufficient) for quasi-symmetric equilibrium, Proposition 4 implies that, if quasi-symmetric equilibrium exists in the pricing game, then it must be unique. Under general conditions, however, it is hard to establish the existence of quasi-symmetric equilibrium. The main reason, as mentioned earlier, is that $Q_1 + Q_2$ may not be logconcave so that the first-order condition (19) may not be sufficient.[21] Nevertheless, in the online appendix we show that in the neighborhood of the candidate equilibrium p^e ((Δ^e, p^e) is the solution to the three-equation system), a small firm's profit function $\pi(p', p^e, \Delta^e)$ is single peaked at $p' = p^e$. Moreover, there are p_1^e and p_2^e, $p_1^e < p^e < p_2^e$, such that the global maximizer of $\pi(p', p^e, \Delta^e)$ must lie in the interval (p_1^e, p_2^e).[22] These two properties imply that $p' = p^e$ is very likely to be the global maximizer of $\pi(p', p^e, \Delta^e)$, or equilibrium exists. When α is big enough so that $\Delta^e \to \overline{x} - \underline{x}$, $p_1^e \to p_2^e$ and thus $p' = p^e$ must be the global maximizer. Therefore, we conclude that the quasi-symmetric equilibrium exists when α

[21] Even when F is uniform, it is hard to analytically check the logconcavity of a small firm's profit in its own price. This is because g and g_N, and hence the integrals, are of complicated form.

[22] See the online appendix for the construction of p_1^e and p_2^e.

is big enough. For some values of α, we numerically verify that, when F is either uniform or truncated normal, $p' = p^e$ is the global maximizer of $\pi(p', p^e, \Delta^e)$ as well.

Note that condition (20) is very similar to the corresponding condition (12) when firm D competes against specialists, except that the coefficient in front of the integral ratio is changed from 1/2 to $\frac{N+1}{2N} > 1/2$. Thus, compared to the model of competing against specialists, under regime DB in the model of competing against generalists, small firms price more aggressively. The underlying reason is as follows. In the model of competing against generalists, for a small firm i, a decrease in p_{iA} not only increases the demand for product iA, but also increases the demand for product iB (firm i's bundle more likely to beat firm D's bundle). The second benefit is absent in the model of competing against specialists, as each firm only produces a single product.

Relatedly, the coefficient $\frac{N+1}{2N}$ decreases in N, indicating that as N increases small firms price less aggressively. The reason for this trend is that a reduction in p_{iA} increases all small firms' demand of product B equally, as iA plus the best product B among all small firms becomes more attractive relative to firm D's bundle. When N gets large, the increase in the demand of iB resulting from a decrease in p_{iA} becomes smaller, since the benefit is diluted among more products B of small firms. Therefore, small firms price less aggressively as N increases.

4.4 Compare three regimes

In this subsection we compare the equilibrium outcomes under three regimes. Denote SS as the regime of separate sales, PB the regime of pure bundling, and DB the regime under which only firm D bundles. We first compare DB and SS. It turns out that the comparison is qualitatively the same as the comparison in the model of competing against specialists.

Proposition 5 *In the model of competing against generalists, parts (i) and (iii) of Proposition 3 hold. Part (ii) of Proposition 3 also holds provided that condition (13) is replaced by the following condition:*

$$\frac{\int_{\underline{x}+\Delta}^{\overline{x}} f(z-\Delta) f_N(z) dz}{\int_{\underline{x}+\Delta}^{\overline{x}} F(z-\Delta) f_N(z) dz} \geq \frac{N+1}{2N} \frac{\int_{\underline{x}+\Delta}^{\overline{x}} g(z-\Delta) g_N(z) dz}{\int_{\underline{x}+\Delta}^{\overline{x}} G(z-\Delta) g_N(z) dz} \tag{21}$$

for $\Delta = \Delta^{SS} \geq \max\{\overline{x} - x_1, x_2 - \underline{x}\}$.

Proposition 5 is almost identical to Proposition 3, except that the coefficient in condition (21) is slightly different form the one in condition (13). The intuition for these results is the same as the one explained in the previous section. It implies that when firm D is sufficiently dominant it will bundle; but if the dominance level α is sufficiently small, it will not.

Next we compare PB and SS. Relative to SS, under PB the distribution of match values changes from F to G for all firms, and correspondingly F_N changes to G_{BN}. Recall that G is a mean-preserving contraction of F. This means that a similar relationship holds between the first-order statistics of N i.i.d. random variables: G_{BN} is less dispersed than F_N. In particular, like Lemma 3, we can show that there exists $x_2 < \overline{x}$, such that for any $x \in (x_2, \overline{x}]$ $g_{BN}(x) < f_N(x)$.

Proposition 6 *Compare PB and SS. (i) If α converges to 0 and N is not too large such that*

$$\int_{\underline{x}}^{\overline{x}} g(z) g_N(z) dz > \int_{\underline{x}}^{\overline{x}} f(z) f_N(z) dz \tag{22}$$

holds (it is satisfied when $N = 1$ (duopoly)), then $P_D^{PB} < p_D^{SS}$, $P^{PB} < p^{SS}$, $\pi_D^{PB} < \pi_D^{SS}$, and $\pi^{PB} < \pi^{SS}$. (ii) If α is big enough such that the equilibrium $\Delta^{SS} \to \overline{x} - \underline{x}$, then (a) $\pi_D^{PB} > \pi_D^{SS}$ and $P_D^{PB} > p_D^{SS}$; (b) $P^{PB} > p^{SS}$ and $\pi^{PB} > \pi^{SS}$.

Proposition 6 shows that, relative to SS, PB makes all firms worse off when the dominance level α is very small (provided that N is not too large), and it makes all firms better off when α is sufficiently large.

The intuition again can be understood in terms of three effects. Since PB leads to thinner tails of the relevant distributions of match values, the demand size effect under PB (relative to SS) largely follows the same pattern as that under DB: when α is small PB reduces firm D's demand, while when α is large PB increases firm D's demand.[23] Regarding the competition between firm D and small firms, because PB also reduces the dispersion of consumers' match values, the marginal consumer effect under PB follows the same pattern as the one under DB: PB increases (decreases) the set of marginal consumers when α is small (large). Finally, the strategic effect now works in the opposite direction under PB: small firms tend to price more aggressively under PB than under SS. In particular, comparing the two pricing equations (6) and (17), we can see that the coefficients in front of the integral ratios are the same under PB and under SS. However, k_B is bigger than k_S (if N is not too large), implying that PB intensifies the competition among small firms.

Taken together, when α is small the marginal consumer effect and the strategic effect dominate: PB intensifies competition and hurts all firms. On the other hand, when α is sufficiently large the marginal consumer effect dominates.[24] This is because in the limit firm D's price is very sensitive to the set of marginal consumers. By reducing the set of marginal

[23] Different from DB, the mix and match effect is absent under PB. This is because under PB consumers are not allowed to mix and match.

[24] When α is large enough, for small firms the competition among themselves is no longer important, compared to the competition between firm D and small firms.

consumers, PB significantly softens competition between firm D and small firms. As a result, all firms benefit.

Now we compare DB and PB for relatively large α, and we focus on small firms, as their incentives dictate which regime will be the equilibrium one. Again, relative to DB, under PB small firms' bundling leads to three effects. We first explain the demand size effect. Specifically, PB changes the relevant distribution from g_N to g_{BN}. Since g_N is the average of two (i.i.d.) first-order statistics of N i.i.d. random variables, while g_{BN} is the 1st-order statistic of N i.i.d. random variables, each being the average of two i.i.d. random variables, the density of g_N has more weights on the right tail than g_{BN}. In other words, G_N dominates G_{BN} in the sense of first-order stochastic dominance. This pattern is illustrated in Figure 5 under the uniform example with $N = 2$, and it is formally stated in the next lemma.

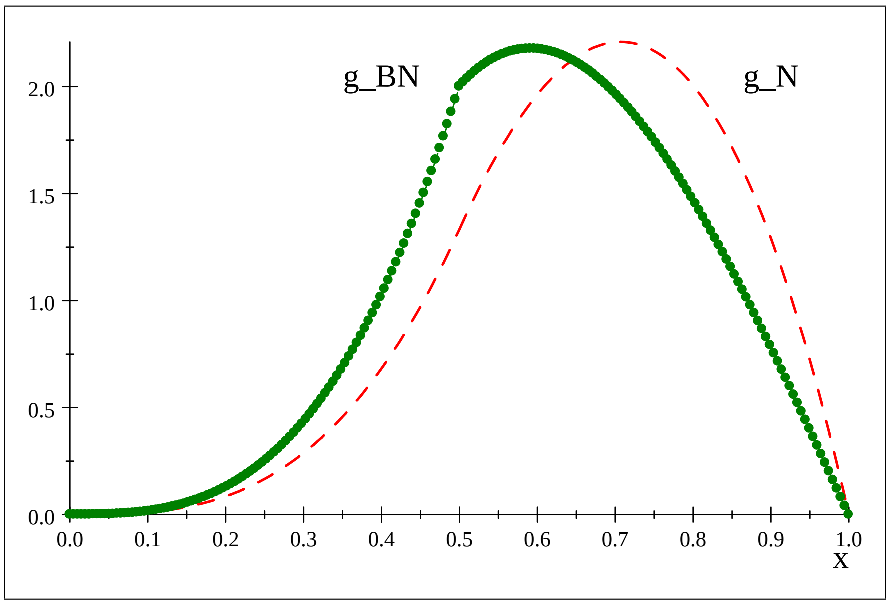

Figure 5: Density functions of G_N and G_{BN}

Lemma 5 *G_N first-order stochastically dominates G_{BN}.*

By Lemma 5, for the same Δ, $\int_{\underline{x}+\Delta}^{\overline{x}} G(x-\Delta)g_N(x)dx > \int_{\underline{x}+\Delta}^{\overline{x}} G(x-\Delta)g_{BN}(x)dx$ always holds. Thus PB tends to reduce small firms' demand. More intuitively, small firms' bundling prevents consumers from mixing and matching among their products, which reduces their demand.[25]

[25] Formally, the difference between G_N and G_{BN} captures the mix and match effect. This is most transparent when $\Delta = 0$. In this case, $\int_{\underline{x}}^{\overline{x}} G(x)g_N(x)dx < \int_{\underline{x}}^{\overline{x}} G(x)g_{BN}(x)dx = N/(N+1) = \int_{\underline{x}}^{\overline{x}} F(x)f_N(x)dx$.

As to the marginal consumer effect, relative to DB, PB tends to reduce the set of marginal consumers between firm D and small firms when α is relatively large. This is again because g_N lies above g_{BN} at the right tail. On the other hand, relative to DB the strategic effect caused by small firms' bundling always intensifies competition, which tends to hurt small firms.[26] Intuitively, under PB a decrease in a small firm's price increases its own demand only, while under DB such a decrease also increases the demand of other small firms.

Proposition 7 *Compare PB and DB. (i) Fix α (not too large) such that the following condition holds*

$$k_B + \frac{\int_{\underline{x}+\Delta}^{\overline{x}} g(z-\Delta)g_{BN}(z)dz}{\int_{\underline{x}+\Delta}^{\overline{x}} G(z-\Delta)g_{BN}(z)dz} > k_S + \frac{N+1}{2N}\frac{\int_{\underline{x}+\Delta}^{\overline{x}} g(z-\Delta)g_N(z)dz}{\int_{\underline{x}+\Delta}^{\overline{x}} G(z-\Delta)g_N(z)dz} \tag{23}$$

for $\Delta = \Delta^{PB}$ (the equilibrium Δ under PB), and the equilibrium $P_D^{PB} \leq P_D^{DB}$, then $p^{PB} < p^{DB}$ and $\pi^{PB} < \pi^{DB}$ (small firms have no incentive to bundle). (ii) If α is big enough such that the equilibrium $\Delta^{DB} \to \overline{x} - \underline{x}$, then $\pi_D^{PB} > \pi_D^{DB}$, $P_D^{PB} > p_D^{DB}$, $p^{PB} > p^{DB}$, $q^{PB} > q^{DB}$, and $\pi^{PB} > \pi^{DB}$ (small firms have incentives to bundle).

Proposition 7 shows that when α is intermediate small firms will not bundle, and they will bundle if α is sufficiently large. Condition (23) in part (i) ensures that, relative to PB, DB softens small firms' competition: $p^{PB}(\Delta^{PB}) < p^{DB}(\Delta^{PB})$. Another condition $P_D^{PB} \leq P_D^{DB}$ also ensures that DB softens firm D's competition (after the equilibrium Δ endogenously adjusts). When α is in an intermediate range, these two conditions are both satisfied in our leading example (F is uniform and $N = 2$).

To understand the results of Proposition 7, we combine the three effects mentioned earlier. When α is intermediate, relative DB, the demand size effect and strategic effect work against small firms under PB. In particular, removing consumers' option of mix and match among small firms' products reduces their demand, while the strategic effect intensifies competition. Thus small firms have no incentive to bundle. When α is sufficiently large, the marginal consumer effect dominates while the other two effects vanish: PB reduces the set of marginal consumers and softens competition. As a result, all firms benefit from PB and small firms will bundle.

Taken together, the equilibrium regime has the following pattern. When the dominance level α is very small, no firm has an incentive to bundle and SS is the equilibrium regime. When α is intermediate, firm D will bundle and small firms will not, thus the equilibrium regime is DB. When α is sufficiently large, all firms will bundle and the equilibrium regime is PB.

[26] More precisely, $k_B > k_S$ if N is not too large; thus PB intensifies competition among small firms. Moreover, the coefficient before the integral ratio in the pricing equations is bigger under PB (1) than under DB ($\frac{N+1}{2N}$), implying small firms price more aggressively under PB.

Using the uniform example with $N = 2$, Figure 6 compares the equilibrium outcomes across three regimes as α changes. It confirms our predicted pattern regarding the relationship between equilibrium regime and the dominance level α.

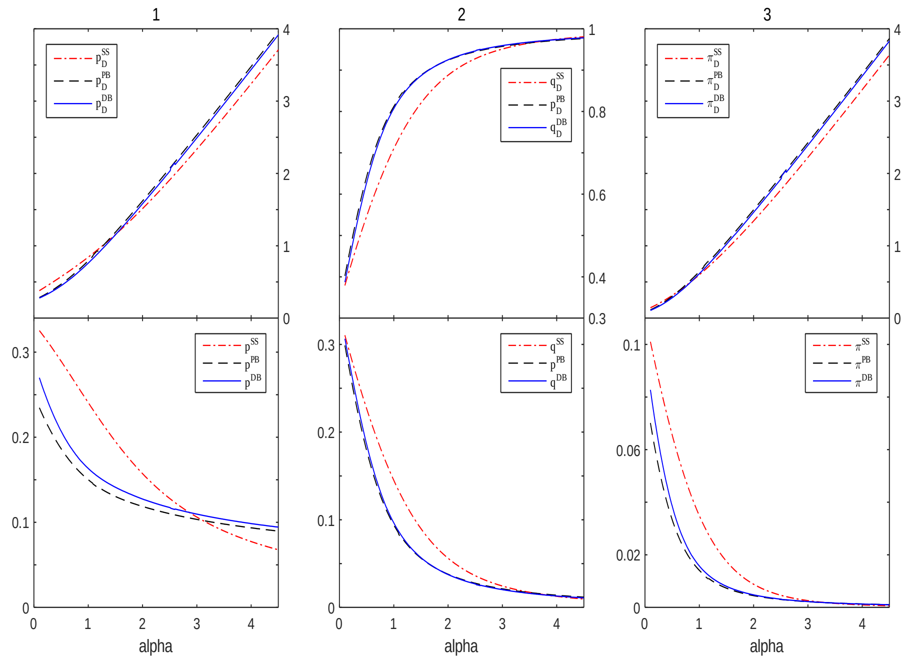

Figure 6: Equilibrium outcomes in the model of competiting with generalists

Specifically, first consider Panel 3 in the figure. Small firms prefer SS to PB when α is smaller than 3.36, and they prefer PB to SS when α is larger than 3.36. On the other hand, firm D prefers SS to DB when α is lower than 0.86, and it prefers DB to SS when α is higher than 0.86. Consequently, we reach the following conclusion regarding the equilibrium regime. (i) SS is the equilibrium regime when α is small (smaller than 0.86); (ii) PB emerges as the equilibrium regime when α is sufficiently large (greater than 3.36); (iii) when the level of dominance α is intermediate (from 0.86 to 3.36), DB is the equilibrium regime.[27] Translating α into firm D's equilibrium market share under SS, it is about 65% when $\alpha = 0.86$ and about 93% when $\alpha = 3.36$. Thus, while the regimes of SS and DB are economically significant, that of PB is economically insignificant: PB is the equilibrium regime only if firm D's equilibrium market share under SS is above 93%, which implies that firm D should be overwhelmingly dominant (effectively monopoly).

[27] Note that, since $N = 2$, given that firm D bundles, a single small firm's bundling means that the other small firm effectively bundles as well, thus changing the regime from DB to PB.

Our results differ qualitatively from those in HJM. Since there are only two firms in their model, HJM only compares SS and PB. In our model with more than two firms, we show that firm D bundles only emerges as the equilibrium regime for a wide range of firm D's dominance level. Moreover, PB in which all firms' bundle is the equilibrium regime only if firm D is overwhelmingly dominant, the case of which is economically insignificant.

Figure 7 illustrates the equilibrium regime as the number of firms and α change. The left panel is the uniform example while in the right panel F is a truncated normal distribution on $[-1, 1]$ with $\sigma = 1$.[28]

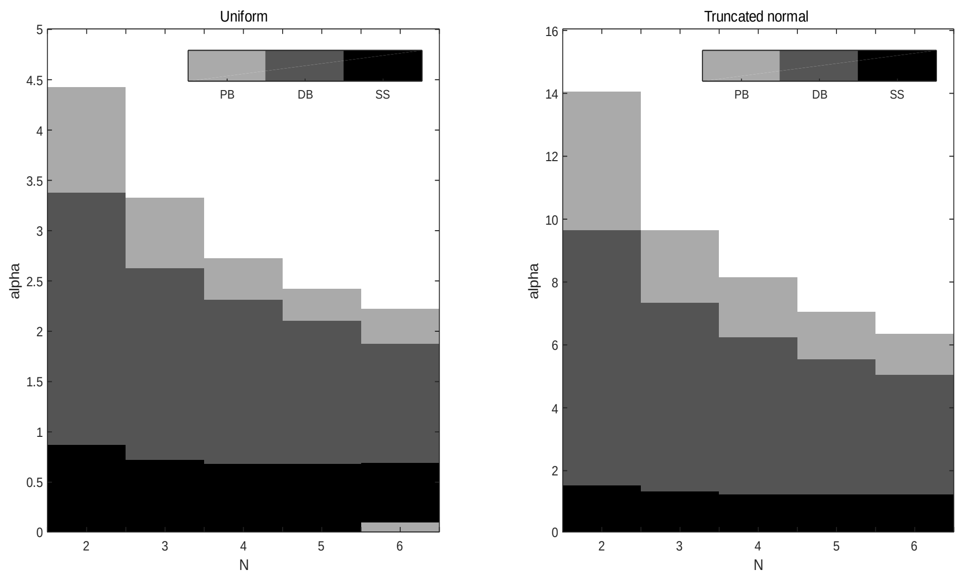

Figure 7: The pattern of equilibrium regimes in the model of competiting with generalists

For any given number of small firms, under both distributions these two panels confirm a common pattern: SS, DB, and PB arise sequentially as the equilibrium regime as the level of dominance α increases.[29] However, we want to point out an exception in the uniform example when $N = 6$: PB emerges as the equilibrium regime when α is very close to zero (all firms are almost symmetric). This region confirms a result in Zhou (2017), who shows that when all firms are symmetric and the number of firms N is large enough, pure bundling by all firms softens competition relative to separate sales.[30] Nevertheless, this region is very small: a little

[28] Again, the upper bound of α is determined by the level at which the equilibrium market share of a small firm diminishes to 0.05% under separate sales.

[29] In the case that $N \geq 3$, for the equilibrium regime to be DB, we aslo need to check that under regime DB a small firm has no incentive to unilaterally deviate to bundling. However, we do not need to worry about such unilateral deviation for the reason mentioned earlier: given a small firm's disadvantage, bundling would reduce its demand significantly as it excludes its products from consumers' choice set of mix and match.

[30] In his example of uniform distribution, the cutoff number of firms is exactly 6.

bit of asymmetry among firms would induce the equilibrium regime to switch from PB to SS.

These two panels also reveal two common patterns when the number of small firms N changes under both distributions. The first pattern is that, as N increases, the cutoff α at which the equilibrium regime switches from DB to PB decreases. This is because with more small firms, each small firm's market share decreases for any given α. Relative to DB, a smaller market share of each small firm reduces the negative impact of the demand size effect caused by PB, which makes PB relatively more attractive than DB for small firms. If we use small firms' equilibrium market share under SS instead of α, the cutoffs between regime DB and regime PB across different Ns are roughly the same (under uniform distribution, as N varies between 2 and 6, the cutoffs market share for individual small firms are within the range of $1.55\% - 1.89\%$).

The second pattern is that, as N increases, the cutoff α at which the equilibrium regime switches from SS to DB decreases slightly. If we use firms D's equilibrium market share under SS instead of α, the cutoff between regime SS and DB decreases considerably as N increases: under uniform distribution, the cutoff market share of firm D changes from 65% (when $N = 2$) to 56% ($N = 6$). This indicates that firm D's incentive to bundle increases with the number of small firms. Note that this pattern is opposite to the pattern in the model of competing with specialists.

To understand the intuition, recall that with more small firms, the advantage of the best possible "bundle" among all small firms' products increases; thus for firm D the loss from being excluded from consumers' choice set of mix and match under DB is magnified, which reduces firm D's incentive to bundle. However, as mentioned earlier (at the end of subsection 4.3), in the model of competing with generalists there is another effect, which is absent in the model of competing against specialists: as N increases small firms compete less aggressively under DB. This (strategic) effect makes DB a more attractive option for firm D as N increases. Overall, our examples indicate that the second effect dominates the first effect.

5 Conclusion

This paper contributes to the literature of competitive bundling by considering a more general market structure in which a multiproduct dominant firm competes with small firms in oligopoly markets. Firms' incentives to bundle depend on both the dominance level and the number of firms, and the dominant firm has a stronger incentive to bundle than do small firms. In the model of competing against specialists, firm D sells separately when the dominance level is low and bundles when the dominance level is high. In the model of competing against generalists, all firms sell separately when the dominance level is low (and the number of firms is not too large), and all firms bundle when the dominance level is very high. When the dominance level is intermediate, a hybrid regime in which the dominant firm bundles, while small firms sell separately, emerges as an equilibrium. Our paper not only offers a more complete analysis of competitive bundling, but also sheds light on the antitrust practice of bundling and tying.

Next, we briefly discuss the impacts of bundling on social welfare. First, since bundling (either only firm D bundles or all firms bundle) restricts consumers' mix and match among firms' products, it tends to reduce social welfare relative to separate sales. Second, under separate sales, firm D's price is higher than small firms', which leads to distortions in product allocation. If bundling reduces the price difference between firm D and small firms so that the market share of firm D increases, it tends to increase social welfare. Otherwise, bundling tends to reduce social welfare. Combining these two effects, we conclude that bundling reduces social welfare if it reduces firm D's market share,[31] and it might increase or reduce social welfare if it increases firm D's market share.

In the model, for simplicity we have assumed that the dominant firm has the same dominance level in both product markets. Here we briefly discuss the case in which firm D has different dominance levels in two markets; that is, $\alpha_A \neq \alpha_B$. First of all, if α_A and α_B are close to each other, then by continuity the results of the baseline model continue to hold qualitatively. Second, when α_A and α_B are sufficiently different, the asymmetry of the dominance levels across two markets will definitely affect firms' incentives to bundle. In particular, it is interesting to ask the following question: Fixing the aggregate level of dominance $\alpha_A + \alpha_B$, how does the asymmetry of the dominance levels affect firms' incentives to bundle? We conjecture that the answer depends on the aggregate level of dominance, and leave this for future research.

[31] This occurs when α is large enough, as bundling will increase the price difference between firm D and small firms.

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Appendix

Proof of Lemma 1.

Proof. The continuity and differentibility of demand in its own price are obvious, given that f is continuous. Thus we only need to show the logconcavity.

We first state several useful properties (Barlow and Prochan, 1975; Bagnoli and Bergstrom, 2005). Suppose X_1 and X_2 are two random variables independent from each other and both have logconcave densities. Then, (a) both $X_1 + X_2$ and $X_1 - X_2$ have logconcave densities, and (b) $\Pr[X_1 > X_2 + v]$ and $\Pr[X_1 < X_2 + v]$ are both logconcave in v.

To show q_D is logconcave in p_D, note that the random variable $\max_{k \neq D}\{X_{kj}\}$ is distributed according to F_N. Its density function is $f_N = NF^{N-1}f$, which is also logconave as the logconcavity of f implies F is also logconcave. Since X_{Dj} and $\max_{k \neq D}\{X_{kj}\}$ are independent and both have logconcave densities, $q_D(p_D, p) = \Pr[X_{Dj} > \max_{k \neq D}\{X_{kj}\} - \alpha - p + p_D]$ is logconcave in p_D. That $q(p_D, p', p)$ is logconcave in p' can be proved in a similar fashion (see a corresponding proof in Lemma 2 for details). ■

Proof of Proposition 1.

Proof. Define the right hand side of p_D in equation (4) as $p_D(\Delta)$, and $p(\Delta)$ is defined accordingly. Define $H(\Delta) = \alpha + p(\Delta) - p_D(\Delta) - \Delta$. The equilibrium Δ^e thus satisfies $H(\Delta^e) = 0$. We first show that $p'(\Delta) < 0$. By (6), it is sufficient to show that the ratio

$$\frac{\int_{\underline{x}+\Delta}^{\overline{x}} f(x-\Delta) f_N(x) dx}{\int_{\underline{x}+\Delta}^{\overline{x}} F(x-\Delta) f_N(x) dx}$$

is increasing in Δ. Define two independent random variables: $\overline{X}_N \sim F_N$ and $X \sim F$. Then $q_S(\Delta) \equiv \int_{\underline{x}+\Delta}^{\overline{x}} F(x-\Delta) f_N(x) dx = \Pr[\overline{X}_N - X > \Delta]$. Since both $\overline{X}_N$ and X have logconcave densities, $q_S(\Delta)$ is logconcave in Δ (properties in the proof of Lemma 1). This implies that

$$\frac{\partial q_S(\Delta)/\partial \Delta}{q_S(\Delta)} = -\frac{\int_{\underline{x}+\Delta}^{\overline{x}} f(x-\Delta) f_N(x) dx}{\int_{\underline{x}+\Delta}^{\overline{x}} F(x-\Delta) f_N(x) dx}$$

is decreasing in Δ. Thus the ratio of the two integrals is increasing in Δ. Therefore, we have $p'(\Delta) < 0$. Similarly, we can show that $p_D(\Delta)$ is increasing in Δ.

Since $p'(\Delta) < 0$ and $p'_D(\Delta) > 0$, $H'(\Delta) < 0$. This implies that $H(\Delta)$ can at most cross 0 once, or equilibrium must be unique.

To show the existence of equilibrium, first consider $H(0)$. When $\Delta = 0$, small firms and firm D become symmetric. Thus $p(0) = p_D(0)$ and $H(0) > 0$. Now consider $H(\alpha)$. Since $p(0) = p_D(0)$ and p is decreasing and p_D is increasing in Δ, we have $H(\alpha) < 0$. By the continuity of $H(\cdot)$, $H(0) > 0$ and $H(\alpha) < 0$ imply that there is a $\Delta^e \in (0, \alpha)$ such that $H(\Delta^e) = 0$. This completes the proof that a quasi-symmetric equilibrium exists and is unique.

Part (i). The equilibrium $\Delta^e \in (0, \alpha)$ has been shown earlier. Since $p(0) = p_D(0)$ and p is decreasing and p_D is increasing in Δ, $\Delta^e > 0$ implies that in equilibrium $p < p_D$. Similarly, $\Delta^e > 0$ also means that in equilibrium $q_D > q$.

Part (ii). As α increases, the curve of $H(\Delta)$ shifts up in a parallel way. Therefore, Δ^e is increasing in α. Since p is decreasing and p_D is increasing in Δ, the equilibrium p is decreasing and the equilibrium p_D is increasing in α. Similarly, in equilibrium q is decreasing and q_D is increasing in α. ■

Proof of Lemma 2.

Proof. Part (i). By properties stated in the proof of Lemma 1, we only need to show that both Z_D and Z_N have logconcave densities (they are independent from each other). Z_D has a logconcave density since X_{DA} and X_{DB} are independent and both have logconcave densities (convolution preserves logconcavity). For Z_N, note that $\overline{X}_{Nj}$'s pdf is $NF^{N-1}f$, which is logconcave. As a result, $Z_N = (\overline{X}_{NA} + \overline{X}_{NB})/2$ has a logconave density as $\overline{X}_{NA}$ and $\overline{X}_{NB}$ are independent and both have logconcave densities.

Part (ii). By equation (8), $q(p', p, P_D)$ is a product of two terms. Thus, to show q is logconcave in p', it is sufficient to show that each term is logconcave in p'. Specifically, the first term is $\Pr[X_{ij} > \max_{k \neq D, i}\{X_{kj}\} - p + p']$. This term is logconave in p', following a similar proof as in Lemma 1. As to the second term, $\Pr[Z_N > Z_D + \alpha - P_D + \frac{p+p'}{2}]$, by a proof similar to part (i), one can show that it is logconave in p'. ■

Proof of Proposition 2.

Proof. The proof largely follows the proof of Proposition 1. In particular, the ratio of the integrals

$$\frac{\int_{\underline{x}+\Delta}^{\overline{x}} g(z-\Delta) g_N(z) dz}{\int_{\underline{x}+\Delta}^{\overline{x}} G(z-\Delta) g_N(z) dz} = -\frac{\partial r(\Delta)/\partial \Delta}{r(\Delta)}, \tag{24}$$

where $r(\Delta) = \int_{\underline{x}+\Delta}^{\overline{x}} G(z-\Delta) g_N(z) dz$. Since both g and g_N are logconcave, $r(\Delta)$ is logconcave in Δ. Thus the ratio of the integrals is increasing in Δ. The monotonicity of $p(\Delta)$, $P_D(\Delta)$, and $H(\Delta)$ also follow, which implies the uniqueness of equilibrium.

However, for the existence of equilibrium, without additional assumptions we are no longer able to show that $H(0) > 0$ and $H(\alpha) < 0$. To show the existence, we thus extend the domain of Δ to the negative region. When $\Delta \le 0$, the pricing equations become

$$\begin{aligned} \frac{1}{P_D} &= \frac{\int_{\underline{x}}^{\overline{x}+\Delta} g(z-\Delta) g_N(z)dz}{\int_{\underline{x}}^{\overline{x}+\Delta}[1-G(z-\Delta)]g_N(z)dz}, \\ \frac{1}{p} &= k_S + \frac{1}{2}\frac{\int_{\underline{x}}^{\overline{x}+\Delta} g(z-\Delta) g_N(z)dz}{1-G_N(\overline{x}+\Delta)+\int_{\underline{x}}^{\overline{x}+\Delta} G(z-\Delta)g_N(z)dz}. \end{aligned}$$

Again, one can check that $p'(\Delta) < 0$, $P_D'(\Delta) > 0$, and $H'(\Delta) < 0$ when $\Delta \le 0$.

We first show that when $\Delta \to \overline{x} - \underline{x}$, $p \to 0$. By (11), it it sufficient to show that the ratio of (24) goes to ∞ in the limit. Let $\Delta = \overline{x} - \underline{x} - \varepsilon$, $\varepsilon > 0$ and $\varepsilon \to 0$. For ε small, the ratio can be approximated by (since both $g(\underline{x}) = 0$ and $g_N(\overline{x}) = 0$, we use $g'(\underline{x})\varepsilon$ and $|g_N'(\overline{x})|\varepsilon$ to approximate the densities)

$$\frac{\int_{\underline{x}+\Delta}^{\overline{x}} g(z-\Delta)g_N(z)dz}{\int_{\underline{x}+\Delta}^{\overline{x}} G(z-\Delta)g_N(z)dz} \simeq \frac{g'(\underline{x})\frac{\varepsilon}{2}|g_N'(\overline{x})|\frac{\varepsilon}{2}\varepsilon}{g'(\underline{x})\frac{\varepsilon^2}{8}|g_N'(\overline{x})|\frac{\varepsilon}{2}\varepsilon} \sim \frac{1}{\varepsilon} \to \infty.$$

Therefore, $\lim_{\Delta \to \overline{x}-\underline{x}} p(\Delta) = 0$. Similarly, we can show that $\lim_{\Delta \to \overline{x}-\underline{x}} P_D(\Delta) \to \infty$, $\lim_{\Delta \to -(\overline{x}-\underline{x})} P_D(\Delta) \to 0$, and $\lim_{\Delta \to -(\overline{x}-\underline{x})} p(\Delta) \to 1/k_S > 0$.

By the above results, we have $H(\overline{x} - \underline{x}) < 0$ and $H(-(\overline{x} - \underline{x})) > 0$. Thus, the continuity of $H(\cdot)$ implies that there is a $\Delta^e \in (-(\overline{x} - \underline{x}), \overline{x} - \underline{x})$ such that $H(\Delta^e) = 0$. This completes the proof that a quasi-symmetric equilibrium exists and is unique.

Different from the case of separate sales, here Δ^e might be greater than α. But if α is large enough, then we can show $\Delta^e < \alpha$. Specifically, define $\widehat{\alpha}$ as $p(\widehat{\alpha}) - P_D(\widehat{\alpha}) = 0$. Such an $\widehat{\alpha} \in (-\overline{x} + \underline{x}, \overline{x} - \underline{x})$ exists since $p(-\overline{x} + \underline{x}) - P_D(-\overline{x} + \underline{x}) > 0$, $p(\overline{x} - \underline{x}) - P_D(\overline{x} - \underline{x}) \to -\infty$, and $p'(\Delta) - P_D'(\Delta) < 0$. When $\alpha > \widehat{\alpha}$, we thus have $H(\alpha) < 0$. Therefore, $\Delta^e < \alpha$, which implies that in equilibrium $p < P_D$. ■

Proof of Lemma 3.

Proof. We first show that x_1 exists. Recall that $f(\underline{x}) > 0$. But $g(\underline{x}) = 0$, since convolution of two i.i.d. random variables leads to 0 density at the boundaries. Thus by continuity $x_1 > \underline{x}$ exists. Next we show x_2 also exists. Recall that $f(\overline{x}) > 0$. But $f_N(\overline{x}) > f(\overline{x})$, since the density of the 1st order statistic of N i.i.d. random variables rotates the density of the original distribution counter-clockwisely. Thus $f_N(\overline{x}) > 0$. On the other hand, $g_N(\overline{x}) = 0$, again because convolution leads to 0 density at the boundaries. Therefore, by continuity, $x_2 < \overline{x}$ exists. ■

Proof of Proposition 3.

Proof. We first explain notations. Δ^{SS} and Δ^{DB} denote equilibrium Δ, while those without these two superscripts are not equilibrium Δ. P_D^{DB} and p_D^{SS} denote equilibrium prices, and other prices denoted differently are not equilibrium prices. Profits and demands are denoted in a similar way.

Part (i). By a similar proof as in the proof of Proposition 2, we can show that as $\Delta^{SS} \to \overline{x} - \underline{x}$, $p^{SS} \to 0$. Since Δ^{SS} is increasing in α, $p^{SS} \to 0$ if α is large enough.

Next we show that if Δ is large enough, then $q_D^{DB}(\Delta) > q_D^{SS}(\Delta)$. Recall (2) and (7)

$$\begin{aligned} q_D^{SS}(\Delta) &= 1 - \int_{\underline{x}+\Delta}^{\overline{x}} F(x-\Delta) f_N(x) dx, \\ q_D^{DB}(\Delta) &= 1 - \int_{\underline{x}+\Delta}^{\overline{x}} G(x-\Delta) g_N(x) dx. \end{aligned}$$

Given the result of Lemma 3, let Δ be big enough such that $\overline{x} - \Delta < x_1$ and $\underline{x} + \Delta > x_2$. Since $\overline{x} - \Delta < x_1$, $f(x-\Delta) > g(x-\Delta)$ for any $x \in [\underline{x}+\Delta, \overline{x}]$, thus $F(x-\Delta) > G(x-\Delta)$ for any $x \in [\underline{x}+\Delta, \overline{x}]$. Similarly, by $\underline{x}+\Delta > x_2$, $g_N(x) < f_N(x)$ for any $x \in [\underline{x}+\Delta, \overline{x}]$. Then we have $\int_{\underline{x}+\Delta}^{\overline{x}} F(x-\Delta) f_N(x) dx > \int_{\underline{x}+\Delta}^{\overline{x}} G(x-\Delta) g_N(x) dx$, as in the two integrals $F(x-\Delta) f_N(x) > G(x-\Delta) g_N(x)$ holds point by point. Thus $q_D^{SS}(\Delta) < q_D^{DB}(\Delta)$ if Δ is large enough.

Now take α large enough such that the equilibrium $\Delta^{SS} \to \overline{x} - \underline{x}$. Recall that in this case $p^{SS} \equiv \epsilon \to 0$. Consider firm D under regime DB. In the worst scenario for firm D, suppose small firms charge $p = 0$. Suppose firm D charges $P_D^{DB\prime} = P_D^{SS} - \epsilon$ so that $\Delta = \Delta^{SS}$ (charging $P_D^{DB\prime}$ may not be firm D's best response when small firms charge 0). In such a scenario, $\pi_D^{DB}(P_D^{SS} - \epsilon, 0) = (P_D^{SS} - \epsilon) q_D^{DB}(\Delta^{SS}) > P_D^{SS} q_D^{SS}(\Delta^{SS}) = \pi_D^{SS}$, since $q_D^{DB}(\Delta^{SS}) > q_D^{SS}(\Delta^{SS})$ and ϵ is small. Therefore, firm D's profit when small firms charge 0 must be higher than π_D^{SS}. This further implies that in the equilibrium under regime DB, firm D's equilibrium profit must be bigger than π_D^{SS}: if small firms charge a positive price, then firm D can increase its price correspondingly but keep $\Delta = \Delta^{SS}$, which leads to a higher profit (a higher price with the same demand $q_D^{DB}(\Delta^{SS})$).

Next we show $P_D^{DB} > p_D^{SS}$. Take an α large enough as in the previous step and fix it. Under this α, suppose the equilibrium $P_D^{DB} \leq p_D^{SS}$. Since $p^{SS} \to 0$, $P_D^{DB} \leq p_D^{SS}$ implies that $\Delta^{DB} \geq \Delta^{SS}$. Because $q_D^{DB}(\Delta)$ is increasing in Δ, we have $q_D^{DB}(\Delta^{SS}) \leq q_D^{DB}(\Delta^{DB})$. By the result in the previous step, $q_D^{SS}(\Delta^{SS}) < q_D^{DB}(\Delta^{SS})$. Therefore, $q_D^{DB}(\Delta^{DB}) > q_D^{SS}(\Delta^{SS})$, or the numerator of P_D^{DB} is bigger than that of p_D^{SS}. Now consider the denominators of p_D^{SS} and P_D^{DB}. By a similar proof as in the earlier step, $\int_{\underline{x}+\Delta^{SS}}^{\overline{x}} f(x-\Delta^{SS}) f_N(x) dx > \int_{\underline{x}+\Delta^{SS}}^{\overline{x}} g(x-\Delta^{SS}) g_N(x) dx$, since the inequality holds point by point for the integrands. Moreover, $\Delta^{DB} \geq \Delta^{SS}$ implies that $\int_{\underline{x}+\Delta^{SS}}^{\overline{x}} g(x-\Delta^{SS}) g_N(x) dx > \int_{\underline{x}+\Delta^{DB}}^{\overline{x}} g(x-\Delta^{DB}) g_N(x) dx$. Thus the denominator of P_D^{DB} is smaller than that of p_D^{SS}. Therefore, $P_D^{DB} > p_D^{SS}$, a contradiction.

Part (ii). Comparing the pricing equations (6) and (12), condition (13) implies that $p^{SS}(\Delta^{SS}) \leq p^{DB}(\Delta^{SS})$. Recall earlier results that both $p^{DB}(\Delta)$ and $p^{SS}(\Delta)$ are decreasing in Δ, and both $P_D^{DB}(\Delta)$ and $p_D^{SS}(\Delta)$ are increasing in Δ.

Now take an α such that $\Delta^{SS} \geq \max\{\overline{x} - x_1, x_2 - \underline{x}\}$. We first show that $P_D^{DB} > p_D^{SS}$. Suppose $P_D^{DB} \leq p_D^{SS}$. It must be the case that $\Delta^{DB} < \Delta^{SS}$. This is because otherwise, by a similar proof as in part (i), we would have $P_D^{DB}(\Delta^{DB}) \geq P_D^{DB}(\Delta^{SS}) > p_D^{SS}(\Delta^{SS})$. Since $p^{DB}(\Delta^{SS}) \geq p^{SS}(\Delta^{SS})$, $\Delta^{DB} < \Delta^{SS}$ and the fact that p^{DB} is decreasing in Δ imply that $p^{DB}(\Delta^{DB}) > p^{SS}(\Delta^{SS})$. But this combining with $P_D^{DB} \leq p_D^{SS}$ means that $\Delta^{DB} > \Delta^{SS}$, a contradiction. Therefore, $P_D^{DB} > p_D^{SS}$.

Next we show that $p^{DB} > p^{SS}$. Suppose $p^{DB} \leq p^{SS}$. Then it must be the case that $\Delta^{DB} \geq \Delta^{SS}$, since otherwise we would have $p^{SS}(\Delta^{SS}) < p^{DB}(\Delta^{DB})$. Combined with the earlier result that $P_D^{DB} > p_D^{SS}$, $\Delta^{DB} \geq \Delta^{SS}$ implies that $p^{DB} > p^{SS}$, a contradiction. Therefore, $p^{DB} > p^{SS}$.

Finally, we show that $\pi_D^{DB} > \pi_D^{SS}$. Under regime DB, suppose firm D charges $P_D' = p^{DB}(\Delta^{DB}) + \alpha - \Delta^{SS}$, with the resulting $\Delta = \Delta^{SS}$. In this case, since $p^{DB} > p^{SS}$ by the earlier result, $P_D' > p_D^{SS}(\Delta^{SS})$. Moreover, by the result in part (i), $q_D^{DB}(\Delta^{SS}) > q_D^{SS}(\Delta^{SS})$. Therefore, firm D's profit $\pi_D^{DB}(P_D', p^{DB}) = P_D' q_D^{DB}(\Delta^{SS}) > p_D^{SS}(\Delta^{SS}) q_D^{SS}(\Delta^{SS}) = \pi_D^{SS}$. Note that P_D' may not be firm D's best response to p^{DB}. This means that firm D's equilibrium profit π_D^{DB} will be weakly higher than $\pi_D^{DB}(P_D', p^{DB})$. Therefore, $\pi_D^{DB} > \pi_D^{SS}$.

Part (iii). Take $\alpha = 0$. Under SS, all firms are symmetric and the equilibrium is symmetric: all firms have the same market share $1/(N+1)$, charge the same price, and $\Delta^{SS} = 0$.

We first show that $q_D^{DB}(\Delta = 0) < q_D^{SS} = 1/(N+1)$. Define $g_{BN} = NG^{N-1}g$. By this notation, $\int_{\underline{x}}^{\overline{x}} G(x) g_{BN}(x)dx = \frac{N}{N+1} \int_{\underline{x}}^{\overline{x}} g_{B(N+1)}(x)dx = N/(N+1)$. By Lemma 5, G_N first-order stochastically dominates G_{BN}. Thus we have $q_D^{DB}(\Delta = 0) = 1 - \int_{\underline{x}}^{\overline{x}} G(x) g_N(x)dx < 1 - \int_{\underline{x}}^{\overline{x}} G(x) g_{BN}(x)dx = 1/(N+1)$. Therefore, $q_D^{DB}(\Delta = 0) < q_D^{SS}$.

Since $q_D^{DB}(0) < q_D^{SS}$, by condition (14), we have $P_D^{DB}(0) < p_D^{SS}$. Next we show that in equilibrium $P_D^{DB} < p_D^{SS}$. Suppose $P_D^{DB} \geq p_D^{SS}$. The condition $p^{DB} < p^{SS}$ implies that $\Delta^{DB} < \Delta^{SS} = 0$. Since $P_D^{DB}(\Delta)$ is increasing in Δ, we have $P_D^{DB} = P_D^{DB}(\Delta^{DB}) < P_D^{DB}(0) < p_D^{SS}$, a contradiction. Thus it must be the case that $P_D^{DB} < p_D^{SS}$.

Next we show that $\pi_D^{DB} < \pi_D^{SS}$. Suppose $\pi_D^{DB} \geq \pi_D^{SS}$. Since $P_D^{DB} < p_D^{SS}$, it must be the case that $q_D^{DB} > q_D^{SS}$. Because $q_D^{DB}(0) < q_D^{SS}$, we must have $\Delta^{DB} > 0$. Now consider firm D under SS. Suppose it charges a price $p_D' = p^{SS} - p^{DB} + P_D^{DB}$. Under this price, $\Delta' = \Delta^{DB}$ and $p_D' > P_D^{DB}$. Firm D's profit under SS becomes

$$\pi(p_D', p^{SS}) = p_D' q_D^{SS}(\Delta^{DB}) > P_D^{DB} q_D^{SS}(\Delta^{DB}) > P_D^{DB} q_D^{DB}(\Delta^{DB}) = \pi_D^{DB} \geq \pi_D^{SS},$$

where the second inequality uses the property that $q_D^{SS}(\Delta) > q_D^{DB}(\Delta)$ for any Δ close to 0. A contradiction. Therefore, it must be the case that $\pi_D^{DB} < \pi_D^{SS}$. ■

Proof of Lemma 4.

Proof. Part (i). For firm D, $q_D(p, P_D)$ is the same as in the model of competing with specialists. The result directly follows Lemma 1.

Part (ii). Both Q_1 and Q_2 are products of two probabilities. Follow a proof similar to Lemma 2, the two terms of either Q_1 or Q_2 are logconcave in p'. Thus both Q_1 and Q_2 are logconave in p'. ■

Proof of Proposition 4.

Proof. The steps are exactly the same as the proof for Proposition 2. The difference between the coefficients in the pricing equations of small firms does not affect the proof. We thus can show that the three-equation system has a unique solution. Since the three-equation system is necessary for equilibrium, we conclude that there is at most one quasi-symmetric equilibrium. ■

Proof of Proposition 5.

Proof. The proof is exactly the same as that of Proposition 3, thus is omitted. ■

Proof of Proposition 6.

Proof. Part (i). Take $\alpha = 0$. Then this becomes the symmetric model in Zhou (2017). Under either regime, in symmetric equilibrium each firm's market share is always $1/(N+1)$. Thus we only need to compare the set of marginal consumers, which is $\int_{\underline{x}}^{\overline{x}} g(z)g_N(z)dz$ under PB and $\int_{\underline{x}}^{\overline{x}} f(z)f_N(z)dz$ under SS. Condition (22) immediately implies that the price and profit of each firm are higher under SS than under PB. By Proposition 1 in Zhou (2017), Condition (22) is always satisfied when $N = 1$, or $\int_{\underline{x}}^{\overline{x}} g^2(z)dz > \int_{\underline{x}}^{\overline{x}} f^2(z)dz$. By continuity, the same results should hold when α is close enough to 0.

Part (iia). It exactly follows the proof of part (i) in Proposition 3, since g_{BN} is also less dispersed than f_N.

Part (iib). Take α big enough such that $\varepsilon = \overline{x} - \underline{x} - \Delta^{PB}$ is very small. Under the same α, let $\varepsilon_S = \overline{x} - \underline{x} - \Delta^{SS}$. We will show that ε_S is of higher order of ε.

By the definition of Δ, we have $\alpha = P_D^{PB} - p^{PB} + \Delta^{PB} = p_D^{SS} - p^{SS} + \Delta^{SS}$. Since in the limit $p \to 0$ and $\Delta \to \overline{x} - \underline{x}$ under both regimes, this implies that $P_D^{PB} \simeq p_D^{SS}$ (both go to ∞). By the pricing equations, this further implies that

$$\lim_{\alpha\to\infty} \frac{\int_{\underline{x}+\Delta^{SS}}^{\overline{x}} f(z-\Delta^{SS})f_N(z)dz}{\int_{\underline{x}+\Delta^{PB}}^{\overline{x}} g(z-\Delta^{PB})g_{BN}(z)dz} \simeq 1.$$

Since ε_S is small, the numerator can be approximated by $f(\underline{x})f_N(\overline{x})\varepsilon_S$. For the denominator, it can be approximated by $g'(\underline{x})\frac{\varepsilon}{2}|g'_{BN}(\overline{x})|\frac{\varepsilon}{2}\varepsilon$. Thus,

$$\lim_{\alpha\to\infty} \frac{\int_{\underline{x}+\Delta^{SS}}^{\overline{x}} f(z-\Delta^{SS})f_N(z)dz}{\int_{\underline{x}+\Delta^{PB}}^{\overline{x}} g(z-\Delta^{PB})g_{BN}(z)dz} \sim \frac{\varepsilon_S}{\varepsilon^3}.$$

Therefore, ε_S is of the same order as ε^3.

Now we show that $q^{PB} > q^{SS}$. It is enough to show that $\int_{\underline{x}+\Delta^{PB}}^{\overline{x}} G(z-\Delta^{PB})g_{BN}(z)dz > \int_{\underline{x}+\Delta^{SS}}^{\overline{x}} F(z-\Delta^{SS})f_N(z)dz$. In particular,

$$\begin{aligned} \int_{\underline{x}+\Delta^{PB}}^{\overline{x}} G(z-\Delta^{PB})g_{BN}(z)dz &\simeq g'(\underline{x})\frac{\varepsilon^2}{8}(-g'_{BN}(\overline{x}))\frac{\varepsilon}{2}\varepsilon \sim \varepsilon^4, \\ \int_{\underline{x}+\Delta^{SS}}^{\overline{x}} F(z-\Delta^{SS})f_N(z)dz &\simeq f(\underline{x})\frac{\varepsilon_S}{2}f_N(\overline{x})\varepsilon_S \sim \varepsilon_S^2 \sim \varepsilon^6. \end{aligned}$$

Since relative to q^{PB}, q^{SS} is of higher order of ε, we have $q^{PB} > q^{SS}$.

Next we show that $P^{PB} > p^{SS}$. By the pricing equations, in the limit

$$\begin{aligned} P^{PB} &\sim \frac{\int_{\underline{x}+\Delta^{PB}}^{\overline{x}} G(z-\Delta^{PB})g_{BN}(z)dz}{\int_{\underline{x}+\Delta^{PB}}^{\overline{x}} g(z-\Delta^{PB})g_{BN}(z)dz} \sim \frac{\varepsilon^4}{\varepsilon^3} \sim \varepsilon, \\ p^{SS} &\sim \frac{\int_{\underline{x}+\Delta^{SS}}^{\overline{x}} F(z-\Delta^{SS})f_N(z)dz}{\int_{\underline{x}+\Delta^{SS}}^{\overline{x}} f(z-\Delta^{SS})f_N(z)dz} \sim \frac{\varepsilon_S^2}{\varepsilon_S} \sim \varepsilon_S. \end{aligned}$$

Since ε_S is of higher order than ε, we have $P^{PB} > p^{SS}$.

Combining $q^{PB} > q^{SS}$ and $P^{PB} > p^{SS}$, we have $\pi^{PB} > \pi^{SS}$. ■

Proof of Lemma 5.

Proof. Define two random variables Y_{BN} and Y_N such that $Y_{BN} \sim G_{BN}$ and $Y_N \sim G_N$. Let y_{BN} and y_N be the realizations of Y_{BN} and Y_N, respectively. Fix any $t \in (\underline{x}, \overline{x})$. We will show that $\Pr[Y_{BN} < t] > \Pr[Y_N < t]$. Suppose $y_N < t$. It must be the case that $y_{BN} < t$. To see this, suppose $y_{BN} \geq t$. It implies that there is an i such that $(x_{iA} + x_{iB})/2 = y_{BN} \geq t$. But iA and iB is also a possible bundle in y_N, thus we have $y_N \geq t$. On the other hand, it is possible that $y_{BN} < t$ but $y_N > t$ (when x_{iA} is the highest among all A products and x_{jB}, $j \neq i$, is the highest among all B products). Therefore, $\Pr[Y_{BN} < t] > \Pr[Y_N < t]$. By definition, $G_{BN}(t) = \Pr[Y_{BN} < t]$ and $G_N(t) = \Pr[Y_N < t]$. Thus, we have $G_{BN}(t) > G_N(t)$ for any $t \in (\underline{x}, \overline{x})$. Therefore, G_N first-order stochastically dominates G_{BN}. ■

Proof of Proposition 7.

Proof. Part (i). Recall that G_N first-order stochastically dominates G_{BN}. Thus we have $q^{PB}(\Delta^{PB}) < q^{DB}(\Delta^{PB})$. By the pricing equations and condition (23), $P^{PB}(\Delta^{PB}) < p^{DB}(\Delta^{PB})$. Next we show that the equilibrium $p^{DB} = p^{DB}(\Delta^{DB}) > P^{PB}$. Suppose to the contrary, $P^{PB} \geq p^{DB}$. But since $P_D^{PB} \leq P_D^{DB}$, we have $\Delta^{DB} \leq \Delta^{PB}$. Because $p^{DB}(\Delta)$ is decreasing in Δ, it implies that $p^{DB}(\Delta^{DB}) \geq p^{DB}(\Delta^{PB}) > P^{PB}(\Delta^{PB})$, a contradiction. Thus, it must be the case that $P^{PB} < p^{DB}$.

Next we show that $\pi^{PB} < \pi^{DB}$. Consider small firms under DB. Suppose all small firms charge a price $p' = P_D^{DB} + \Delta^{PB}$. Since $P_D^{PB} \leq P_D^{DB}$, $p' \geq P^{PB}$. Under this price, small firms' demand equals to $q^{DB}(\Delta^{PB}) > q^{PB}(\Delta^{PB})$. Thus, $\pi^{DB}(p', P_D^{DB}) = p' q^{DB}(\Delta^{PB}) > P^{PB} q^{PB}(\Delta^{PB}) = \pi^{PB}$. Note that this price p' is not necessarily small firms' best response to P_D^{DB}. This means that π^{DB} is weakly higher than $\pi^{DB}(p', P_D^{DB})$. Therefore, $\pi^{PB} < \pi^{DB}$.

Part (ii). Take α big enough such that $\varepsilon = \overline{x} - \underline{x} - \Delta^{DB}$ is very small. Under the same α, let $\varepsilon_B = \overline{x} - \underline{x} - \Delta^{PB}$. We will investigate the relationship between ε and ε_B.

In the equilibrium conditions, since P_D goes to infinity and it is very sensitive to Δ, we must have $\int_{\underline{x}+\Delta^{PB}}^{\overline{x}} g(z-\Delta^{PB}) g_{BN}(z) dz = \int_{\underline{x}+\Delta^{DB}}^{\overline{x}} g(z-\Delta^{DB}) g_N(z) dz$. This equality can be approximated as

$$g'(\underline{x}) \frac{\varepsilon_B}{2} |g'_{BN}(\overline{x})| \frac{\varepsilon_B}{2} \varepsilon_B = g'(\underline{x}) \frac{\varepsilon}{2} |g'_N(\overline{x})| \frac{\varepsilon}{2} \varepsilon.$$

Thus $\varepsilon_B = (\frac{|g'_N(\overline{x})|}{|g'_{BN}(\overline{x})|})^{\frac{1}{3}} \varepsilon$. Since $g_N(\overline{x}) = g_{BN}(\overline{x}) = 0$ and $g_N(x) > g_{BN}(x)$ in the neighborhood of $\overline{x}$, it must be the case that $|g'_N(\overline{x})| > |g'_{BN}(\overline{x})|$. Therefore, $\varepsilon_B > \varepsilon$.

Now we compare the market shares. Specifically,

$$\begin{aligned} Nq^{DB} &= \int_{\underline{x}+\Delta^{DB}}^{\overline{x}} G(z-\Delta^{DB}) g_N(z) dz \simeq g'(\underline{x}) |g'_N(\overline{x})| \frac{\varepsilon^4}{16}, \\ Nq^{PB} &= \int_{\underline{x}+\Delta^{PB}}^{\overline{x}} G(z-\Delta^{PB}) g_{BN}(z) dz \simeq g'(\underline{x}) |g'_{BN}(\overline{x})| \frac{\varepsilon_B^4}{16}, \\ \frac{q^{PB}}{q^{DB}} &= \frac{|g'_{BN}(\overline{x})| \varepsilon_B^4}{|g'_N(\overline{x})| \varepsilon^4} = \frac{\varepsilon_B}{\varepsilon} > 1. \end{aligned}$$

Therefore, $q^{PB} > q^{DB}$.

Next we show that $P^{PB} > p^{DB}$. By the pricing equations, in the limit

$$\begin{aligned} P^{PB} &\simeq \frac{\int_{\underline{x}+\Delta^{PB}}^{\overline{x}} G(z-\Delta^{PB}) g_{BN}(z) dz}{\int_{\underline{x}+\Delta^{PB}}^{\overline{x}} g(z-\Delta^{PB}) g_{BN}(z) dz} \simeq \frac{\varepsilon_B^4/16}{\varepsilon_B^3/4} \simeq \varepsilon_B/4, \\ p^{DB} &\simeq \frac{\int_{\underline{x}+\Delta^{DB}}^{\overline{x}} G(z-\Delta^{DB}) g_N(z) dz}{\int_{\underline{x}+\Delta^{DB}}^{\overline{x}} g(z-\Delta^{DB}) g_N(z) dz} \simeq \frac{\varepsilon^4/16}{\varepsilon^3/4} \simeq \varepsilon/4. \end{aligned}$$

Since $\varepsilon_B > \varepsilon$, we have $P^{PB} > p^{DB}$.

Combining $q^{PB} > q^{DB}$ and $P^{PB} > p^{DB}$, we get the desired result $\pi^{PB} > \pi^{DB}$.

Since $P^{PB} > p^{DB}$, and for any Δ the inequality $q_D^{BB}(\Delta) > q_D^{DB}(\Delta)$ holds, by a similar argument as in the proof of Proposition 3, we can show that $\pi_D^{PB} > \pi_D^{DB}$ and $P_D^{PB} > p_D^{DB}$. ∎